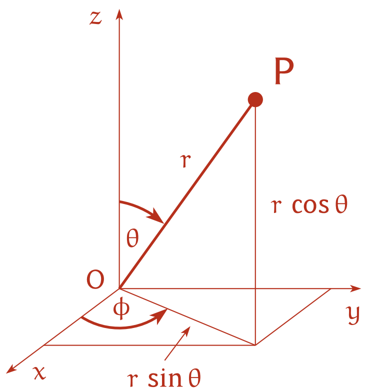
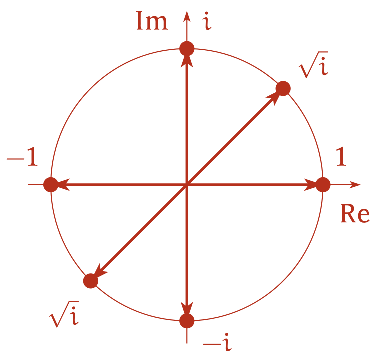
Math Basics for the Introductory Courses in Physics

for University Students of Physics as Main or Side Subject





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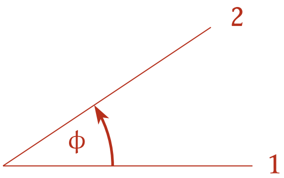
1 Angles and Coordinate Systems

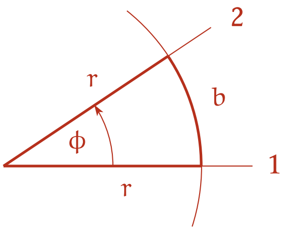


We consider only angles in the plane. A **plane angle** characterizes the relative rotation of two semi-infinite straight lines (→ **legs**) crossing in a point (→ **apex**), so it measures the **relative orientation** of the two lines. Angles are usually denoted by Greek characters.

Coordinates and **coordinate systems** fulfill the purpose of unequivocally defining the positions of points on a plane, in the three-dimensional (3-d) space or in a space of higher dimension (n-d). There are many possibilities of defining coordinates. We discuss the most important ones.

1.1 Plane Angles

1.1.1 Magnitude of an Angle

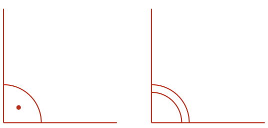


- Draw a circle with radius r around the apex. The legs cut an arc of length b out of it. The magnitude of the corresponding angle ϕ (in radians) is then defined as

$$\phi = \frac{\text{arc length}}{\text{radius}} = \frac{b}{r} \tag{1.1}$$

- One defines

full angle,	$\phi = 2\pi$
reflex angle,	$\pi < \phi < 2\pi$
straight angle,	$\phi = \pi$
obtuse angle,	$\pi/2 < \phi < \pi$
right angle,	$\phi = \pi/2$
acute angle,	$0 < \phi < \pi/2$



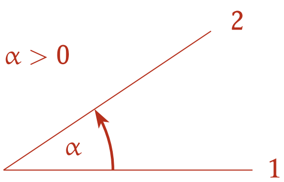
- Right angles are often indicated by a **dot** or a **double arc**.
- Unit of angles

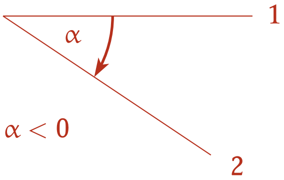
$$[\phi] = \frac{1\ \text{m}}{1\ \text{m}} = 1\ \text{radian (rad)}$$

In addition: Division in degrees (°), arc minutes (′), and arc seconds (″),

$$1^\circ \mathrel{\widehat{=}} 60' \mathrel{\widehat{=}} 3600''$$

A full angle has the magnitude 360° or 2π (rad), respectively.





- **Sense of rotation, sign of angles:** A **counter-clockwise** rotation is defined to have **positive** sign, a **clockwise** one is **negative**. This convention applies also to angles measuring the rotation.

1.1.2 Converting between Degrees and Radians

- Radians → degrees, minutes, seconds

$$\phi = x \text{ rad} = x \text{ rad} \frac{360^\circ}{2\pi \text{ rad}} = x \left(\frac{180}{\pi}\right)^\circ \quad (1.2)$$

- Degrees, minutes, seconds → radians

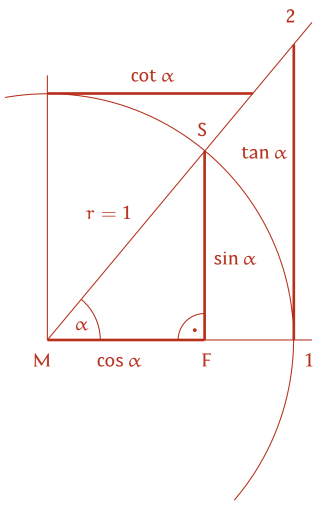
π rad $\widehat{=}$ 180°

$$\psi = y^\circ = y^\circ \frac{2\pi \text{ rad}}{360^\circ} = y \left(\frac{\pi}{180}\right) \text{ rad} \quad (1.3)$$

- Examples

$$1 \text{ rad} \widehat{=} 57{,}2958^\circ = 57^\circ 17' 45''$$
$$1^\circ \widehat{=} 1{,}7453 \cdot 10^{-2} \text{ rad} = 0{,}017453 \text{ rad} = 17{,}453 \text{ mrad}$$
$$(1 \text{ mrad} = 10^{-3} \text{ rad} \quad [\text{milli-radians}])$$

1.1.3 Trigonometric Funktions



- Draw a circle with radius r around an apex M. Two semi-infinite straights 1 and 2 enclose the angle α. From the intersection S between straight 2 and the circle, draw a line perpendicular to straight 1, hitting it in base point F. MFS is then a **right triangle** having the circle radius r as hypotenuse. Consider also the vertical tangent to the circle on the right and the horizontal tangent at the top. The figure shows the situation for $r = 1$ ("unit circle").

- The **trigonometric functions** of the angle α are defined as follows

$$\sin\alpha = \frac{\overline{FS}}{\overline{MS}} \quad (1.4)$$

$$\cos\alpha = \frac{\overline{MF}}{\overline{MS}} \quad (1.5)$$

$$\tan\alpha = \frac{\overline{FS}}{\overline{MF}} = \frac{\sin\alpha}{\cos\alpha} \quad (1.6)$$

$$\cot\alpha = \frac{\overline{MF}}{\overline{FS}} = \frac{\cos\alpha}{\sin\alpha} = \frac{1}{\tan\alpha} \quad (1.7)$$

→ Chapter 3

More about these functions follows in Chapter 3.

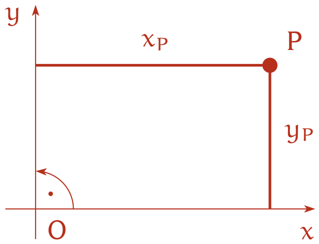
- If we know, e. g., the value of $\tan\alpha$, how can we obtain the angle α? This is possible via the **arc functions** or **inverse trigonometric functions**. Example

$$\alpha = \arctan(\tan\alpha) = \arctan\left(\frac{\overline{FS}}{\overline{MF}}\right) \quad (1.8)$$

They are defined to all the trigonometric functions. More details in Chapter 3.

1.2 Coordinate Systems in the Plane

1.2.1 Cartesian Coordinates



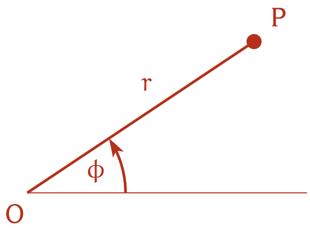
- A point in the plane is defined as **coordinate origin**. Through this point, two mutually perpendicular straigh lines are laid, the **coordinate axes**. Then any point in the plane can be uniquely characterized by its smallest distances (with signs!) from the axes, its **Cartesian coordinates**,

$$P = (x_P; y_P) \tag{1.9}$$

- Other names for the x axis are 1-axis or **abscissa**, other words for the y axis are 2-axis or **ordinate**.
- The angle from the positive x or 1-axis to the positive y or 2-axis must be **positive**.

1.2.2 Polar Coordinates

- This is a coordinate system not using straight axes.
- Define a point as **coordinate origin** and **one reference axis**. The latter usually coincides with the x axis (or 1-axis) of a Cartesian system.

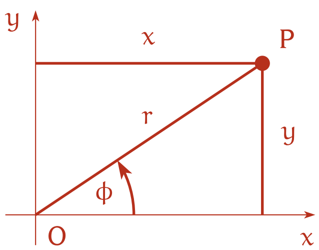


- The position of a point P is characterized by its **distance from the origin** and the **angle** between the connecting line $\overline{OP}$ and the reference axis,

$$P = (r; \phi) \tag{1.10}$$

- The **coordinate transformation** is easily possible with the trigonometric functions and elementary geometry. (The index "P" of the Cartesian coordinates is omitted in the following.)

$$P = (x; y) = (r; \phi) \tag{1.11}$$



so

$$x = r\,\cos\phi \tag{1.12}$$
$$y = r\,\sin\phi \tag{1.13}$$

and, in the other direction[1],

$$r = \sqrt{x^2 + y^2} \qquad \text{(Pythagoras's law)} \tag{1.14}$$
$$\phi = \arctan\frac{y}{x} \qquad \text{bzw.} \quad \tan\phi = \frac{y}{x} \tag{1.15}$$

[1] In this and the following transformations for angle coordinates, add π to the arctan function in the second and third quadrant, e. g., $\phi = \arctan(y/x) + \pi$.

1.3 Coordinate Systems in 3-d Space

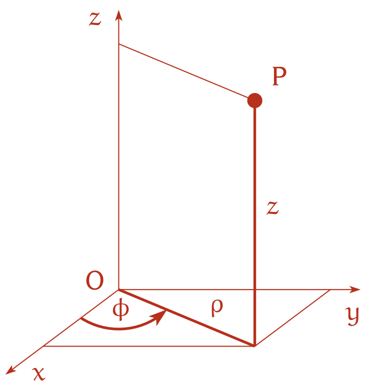
1.3.1 Cartesian Coordinates

corkscrew rule

Perpendicular to the xy plane, a third coordinate axis (z axis or 3-axis) is added such that x, y, and z axis form a **right-handed coordinate system**, i. e., the three axes are represented by thumb, index-, and middle finger of the **right** hand (in this order!). The **Cartesian coordinates** of any point in 3-d space are then its shortest distances (again with sign!) from the planes spanned by each two of the axes,

x coordinate	—	distance from the yz plane
y coordinate	—	distance from the zx plane
z coordinate	—	distance from the xy plane

1.3.2 Cylindrical Coordinates



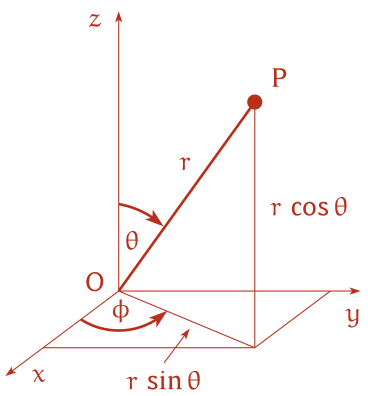
- Starting from polar coordinates in the plane, add the perpendicular z axis through the coordinate origin. Any point in 3-d space is then uniquely defined by **two length coordinates** (ρ and z) and **one angle** (ϕ).
- The distance from the origin in the xy plane is often called ρ (rather than r) in order not to confuse it with spherical coordinates (see below).
- All the points with $\rho = \text{const.}$ are located on the **surface shell of a cylinder**.
- Coordinate transformation

$$x = \rho \cos \phi \tag{1.16}$$
$$y = \rho \sin \phi \tag{1.17}$$
$$z = z \tag{1.18}$$

and, in the other direction,

$$\rho = \sqrt{x^2 + y^2} \tag{1.19}$$
$$\phi = \arctan \frac{y}{x} \qquad \text{or} \qquad \tan \phi = \frac{y}{x} \tag{1.20}$$
$$z = z \tag{1.21}$$

1.3.3 Spherical Coordinates



- Here, the position of a point P is defined by its **distance r from the origin** and **two angles**.
- θ is the angle between the connecting line $\overline{OP}$ and the positive z axis.
- ϕ is the angle between the projection of $\overline{OP}$ onto the xy plane and the positive x axis. The projection is equivalent to the cylindrical coordinate ρ; it has the length $r \sin \theta = \sqrt{x^2 + y^2}$. Consequently, ϕ is identical in cylindrical and spherical coordinates.
- θ varies between 0 and π, ϕ between 0 and 2π. θ is called **polar angle**, ϕ **azimuth (angle)**.

- All the points with $r = \text{const.}$ are located on the **surface of a sphere**. The xy plane is its equatorial plane, the point $\theta = 0$ (i. e., $x = y = 0$; $z = r$) its north pole, and the point $\theta = \pi$ ($x = y = 0$; $z = -r$) its south pole. The azimuth ϕ corresponds to the geographical longitude, $\pi/2 - \theta$ to the latitude.
- Coordinate transformation

$$x = r \sin\theta \cos\phi \tag{1.22}$$
$$y = r \sin\theta \sin\phi \tag{1.23}$$
$$z = r \cos\theta \tag{1.24}$$

and, the other way around,

$$r = \sqrt{x^2 + y^2 + z^2} \tag{1.25}$$
$$\theta = \arctan \frac{\sqrt{x^2 + y^2}}{z} \qquad \text{or} \qquad \tan\theta = \frac{\sqrt{x^2 + y^2}}{z} \tag{1.26}$$
$$\phi = \arctan \frac{y}{x} \qquad \text{or} \qquad \tan\phi = \frac{y}{x} \tag{1.27}$$

2 Vector Analysis

Some quantities in physics are not completely characterized by their **value** alone; instead, a **direction** must also be given. These quantities are dubbed **vectors;** examples include velocity, force, or momentum. Many laws in physics can be formulated in a simpler and more elegant way with vector analysis.

2.1 Scalars and Vectors

- **Scalar:** Quantity without a direction, characterized by **value** · **unit**. Examples:

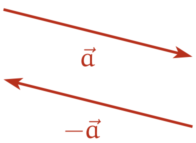
time	$t = 10$ s
mass	$m = 50$ kg
volume	$V = 120$ m^3
temperature	$T = 298$ K

- **Vector:** Quantity which also needs its **direction** to be specified for complete characterization; hence, **value** · **unit plus indication of the direction**. The name "vactor" derives from astronomy: It denotes the fictitious straight line drawn from the Sun to a planet, i. e., from one of the foci to a point on the ellipsoidal orbit of the planet. An example is the velocity: $\vec{v} = 20$ m/s, with additional indication of the direction in which a particle moves. Other examples of vectors are momentum $\vec{p}$, acceleration $\vec{a}$, angular momentum $\vec{L}$, electrical and magnetic field strength ($\vec{E}$ and $\vec{H}$, respectively). Notion (e. g., for a velocity vector): $\xrightarrow{\vec{v}}$. In print, vectors are often indicated by bold-face characters: $\xrightarrow{\mathbf{v}}$.

$\xrightarrow{\vec{v}}$ $\xrightarrow{\mathbf{v}}$

- **Absolute value** of a vector: $|\vec{v}| = v$. The absolute value of a vector (its "length") is a scalar.

- **Zero vector:** The vector with value zero is the zero vector. Its direction is undefined. Notion: $\vec{0}$.



- **Negative Vector:** Start with a given vector $\vec{a}$. The negative vector $-\vec{a}$ has the same absolute value as $\vec{a}$ but the **opposite direction**.

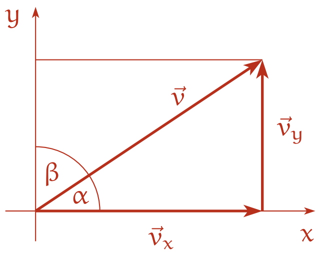
- **Unit Vector:** $\vec{v}/|\vec{v}| = \vec{e}_v = \hat{e}_v$. The unit vector in a given direction has the absolute value one (or unity). Unit vectors are often denoted with a hat (^) instead of an arrow. Then

$$|\hat{e}_v| = \left|\frac{\vec{v}}{|\vec{v}|}\right| = \frac{|\vec{v}|}{|\vec{v}|} = 1 \tag{2.1}$$

$$\vec{v} = |\vec{v}| \cdot \hat{e}_v = v \cdot \hat{e}_v \tag{2.2}$$

$\vec{v} = |\vec{v}| \cdot \hat{e}_v$

(v: value with measurement unit, $\hat{e}_v$: unit vector in the direction of $\vec{v}$).



- **Components of a Vector:** The components of a vector are its projections onto the axes of a coordinate system. For a 2-d vector in the plane, e. g., this reads

$$\vec{v} = \vec{v}_x + \vec{v}_y = |\vec{v}_x| \cdot \hat{e}_x + |\vec{v}_y| \cdot \hat{e}_y = v_x \cdot \hat{e}_x + v_y \cdot \hat{e}_y \tag{2.3}$$

The transition to three dimensions is straightforward,

$$\vec{v} = v_x \cdot \hat{e}_x + v_y \cdot \hat{e}_y + v_z \cdot \hat{e}_z \tag{2.4}$$

v_x, v_y, v_z are the components of the vector; $\hat{e}_x$, $\hat{e}_y$, $\hat{e}_z$ are the unit vectors parallel to the coordinate axes.

- **Parallel Shift:** You can apply a parallel shift to an arrow representing a vector. This does not change the vector!

- **Representation with Components:** With the unit vectors

$$\hat{e}_x = \begin{pmatrix}1\\0\\0\end{pmatrix} ; \quad \hat{e}_y = \begin{pmatrix}0\\1\\0\end{pmatrix} ; \quad \hat{e}_z = \begin{pmatrix}0\\0\\1\end{pmatrix} \tag{2.5}$$

an arbitrary vector $\vec{v}$ can be written as

$$\begin{aligned}\vec{v} &= v_x \cdot \hat{e}_x + v_y \cdot \hat{e}_y + v_z \cdot \hat{e}_z \\ &= v_x \begin{pmatrix}1\\0\\0\end{pmatrix} + v_y \begin{pmatrix}0\\1\\0\end{pmatrix} + v_z \begin{pmatrix}0\\0\\1\end{pmatrix} \\ &= \begin{pmatrix}v_x\\v_y\\v_z\end{pmatrix}\end{aligned} \tag{2.6}$$

- **Absolute Value of a Vector:** For the 2-d example above we can write

$$|\vec{v}_x| = v_x = v \cdot \cos\alpha \tag{2.7}$$
$$|\vec{v}_y| = v_y = v \cdot \cos\beta = v \cdot \sin\alpha \tag{2.8}$$

and, with Pythagoras's law,

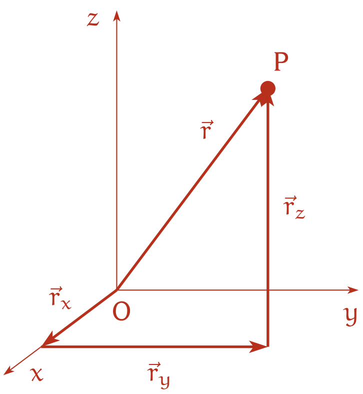
$$v_x^2 + v_y^2 = v^2 \cdot \left(\cos^2\alpha + \sin^2\alpha\right) = v^2 \tag{2.9}$$

thus

$$v = \sqrt{v_x^2 + v_y^2} \tag{2.10}$$

The 3-d analog reads

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} \tag{2.11}$$



- **Position Vector:** The position of a point P in space can be defined by a vector from the origin O of the coordinate system to this point P,

$$\begin{aligned}\vec{r} &= \vec{r}_x + \vec{r}_y + \vec{r}_z \\ &= x \cdot \hat{e}_x + y \cdot \hat{e}_y + z \cdot \hat{e}_z = \begin{pmatrix}x\\y\\z\end{pmatrix}\end{aligned} \tag{2.12}$$

$r = \sqrt{x^2 + y^2 + z^2}$

with

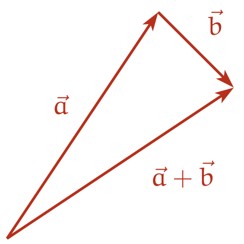
$$|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2} \tag{2.13}$$

Example

$$\vec{r} = 2\ \text{cm} \cdot \hat{e}_x - 3\ \text{cm} \cdot \hat{e}_y + 1\ \text{cm} \cdot \hat{e}_z$$
$$r = \sqrt{14}\ \text{cm} \approx 3{,}74\ \text{cm}$$

2.2 Calculating with Vectors

2.2.1 Vector Addition



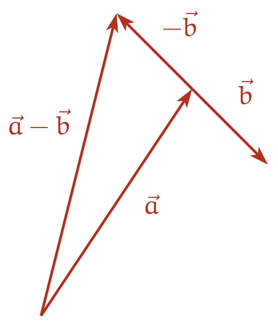
Start with a vector $\vec{a}$ and draw a second vector $\vec{b}$ from its end point. Then the **sum vector** $\vec{c} = \vec{a} + \vec{b}$ is the vector connecting the starting point of $\vec{a}$ with the end point of $\vec{b}$. In a similar way this applies to more than two vectors.

$$\vec{c} = \vec{a} + \vec{b} = \vec{b} + \vec{a} \tag{2.14}$$

with

$$\begin{pmatrix} c_x \\ c_y \\ c_z \end{pmatrix} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} + \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} a_x + b_x \\ a_y + b_y \\ a_z + b_z \end{pmatrix} \tag{2.15}$$

2.2.2 Vector Subtraction



Subtracting a vector $\vec{b}$ means **adding the negative vector** $-\vec{b}$.

$$\vec{d} = \vec{a} - \vec{b} = \vec{a} + \left(-\vec{b}\right) \tag{2.16}$$

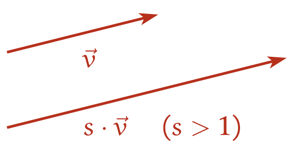
with

$$\begin{pmatrix} d_x \\ d_y \\ d_z \end{pmatrix} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} - \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} a_x - b_x \\ a_y - b_y \\ a_z - b_z \end{pmatrix} \tag{2.17}$$

Example

$$\vec{a} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}; \quad \vec{b} = \begin{pmatrix} -3 \\ 1 \\ -7 \end{pmatrix} \quad \hookrightarrow \quad \vec{a} - \vec{b} = \begin{pmatrix} 8 \\ -3 \\ 8 \end{pmatrix}$$

2.2.3 Multiplication of a Vector with a Scalar



Let $\vec{v}$ be a vector and s a scalar. Then the vector $s \cdot \vec{v}$ has the $|s|$-fold amount of $\vec{v}$. It points in the same direction as $\vec{v}$ for $s > 0$ and in the opposite direction for $s < 0$.

$$s \cdot \vec{v} = s \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} s \cdot v_x \\ s \cdot v_y \\ s \cdot v_z \end{pmatrix} \tag{2.18}$$

The **distributive law** applies,

$$s \cdot (\vec{v} + \vec{w}) = s \cdot \vec{v} + s \cdot \vec{w} \tag{2.19}$$
$$(s + t) \cdot \vec{v} = s \cdot \vec{v} + t \cdot \vec{v} \tag{2.20}$$

2.2.4 Scalar Product (Dot Product, Inner Product, Direct Product)

- **Definition**

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z = a\, b\, \cos\alpha \tag{2.21}$$

where α is the angle enclosed by the vectors $\vec{a}$ and $\vec{b}$. Important: This formula can be used to calculate angles!

$\cos\alpha = \dfrac{\vec{a} \cdot \vec{b}}{a\, b}$

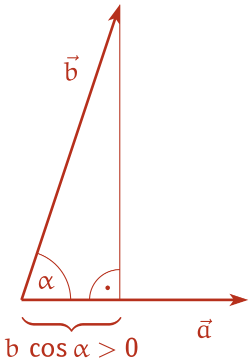
$$\boxed{\cos\alpha = \frac{\vec{a} \cdot \vec{b}}{a\, b} = \frac{a_x b_x + a_y b_y + a_z b_z}{a\, b}} \tag{2.22}$$

- The following rules apply:
Commutative law

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \tag{2.23}$$

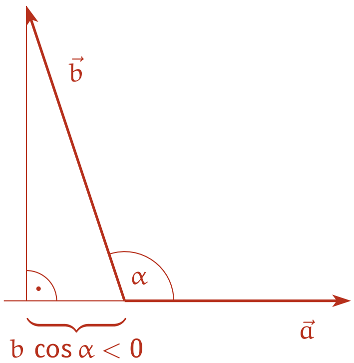
Distributive law

$$\vec{a} \cdot \left(\vec{b} + \vec{c}\right) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} \tag{2.24}$$



- If two vectors ($\neq \vec{0}$) are **perpendicular to each other**, their scalar product yields **zero**,

$$\vec{a} \cdot \vec{b} = 0 \quad \leftrightarrow \quad \vec{a} \perp \vec{b} \tag{2.25}$$

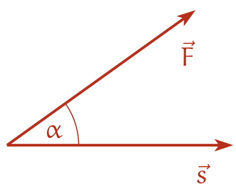


- **Geometrical Interpretation:** The scalar product is the product of the absolute value of one vector with the projection of the second vector onto the direction of the first one. Note: This "projection" can be positive or negative, depending on the size of α (positive for $\alpha < 90°$, negative for $\alpha > 90°$).

- Numerical example

$$\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}; \quad \vec{b} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}; \quad \hookrightarrow \quad \vec{a} \cdot \vec{b} = 6$$

$$\cos\alpha = \frac{6}{3\sqrt{6}} = \frac{2}{\sqrt{6}}; \qquad \text{so} \quad \alpha = 35{,}26°$$



- Example from physics: A force $\vec{F}$ acts on a mass, which is moving along a path $\vec{s}$. Then the **work** performed by the force on the mass is equal to

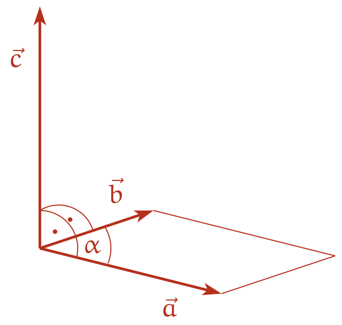
$$W = \vec{F} \cdot \vec{s}$$

If $\vec{F}$ and $\vec{s}$ are **perpendicular to each other**, the work is zero.

2.2.5 Vector Product (Cross Product, Outer Product)

- **Definition:** The vector product $\vec{a} \times \vec{b} = \vec{c}$ is a vector whose direction is perpendicular to $\vec{a}$ and $\vec{b}$ and whose absolute value is equal to the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$,

$$|\vec{c}| = c = a\, b\, \sin\alpha. \tag{2.26}$$



$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

- The vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ (in this order!) compose a **right-handed trihedron**, i. e., they are represented by thumb, index-, and middle finger (also in this order!) of the **right** hand.
- The **distributive law** applies,

$$\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}. \tag{2.27}$$

- The **commutative law** is **not valid** here,

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}. \tag{2.28}$$

Reason: the angle α (and also its sine value) changes sign when $\vec{a}$ and $\vec{b}$ are exchanged.

- If two vectors ($\neq \vec{0}$) are **parallel or anti-parallel** to each other, their vector product yields **zero**,

$$\vec{a} \times \vec{b} = \vec{0} \quad \leftrightarrow \quad \vec{a} \parallel \vec{b}. \tag{2.29}$$

Especially the vector product of any vector with itself yields zero,

$$\vec{a} \times \vec{a} = \vec{0}. \tag{2.30}$$

- The components of the vector product are calculated as follows.

$$\vec{c} = \vec{a} \times \vec{b} \tag{2.31}$$

with

$$\begin{pmatrix} c_x \\ c_y \\ c_z \end{pmatrix} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \times \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} a_y\, b_z - a_z\, b_y \\ a_z\, b_x - a_x\, b_z \\ a_x\, b_y - a_y\, b_x \end{pmatrix} \tag{2.32}$$

- The following 3-d determinant yields the same result

$$\vec{c} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \tag{2.33}$$

Evaluating a determinant means subtracting the sum of the "left-hand diagonal products" from that of the "right-hand diagonal products".

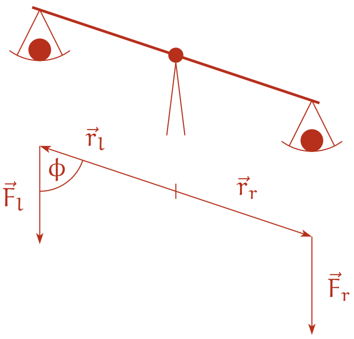
- Numerical example

$$\vec{a} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}; \quad \vec{b} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \quad \hookrightarrow$$

$$\vec{a} \times \vec{b} = \begin{pmatrix} 4-2 \\ -3-8 \\ -8-6 \end{pmatrix} = \begin{pmatrix} 2 \\ -11 \\ -14 \end{pmatrix}$$

- Practical application: Let us assume that a force $\vec{F}$ pulls on a lever arm $\vec{r}$ (e. g., the weight of a mass on a scale beam). The resulting **torque** is

$$\vec{M} = \vec{r} \times \vec{F}$$



The vector of the torque points **along the axis of the balance**. The balance is in equilibrium, if the torques exerted by the two weights have the same absolute value and point in opposite directions,

$$\left|\vec{M}_l\right| = \left|\vec{M}_r\right|$$
$$r_l\, F_l\, \sin\phi = r_r\, F_r\, \sin(\pi - \phi) = r_r\, F_r\, \sin\phi$$
$$r_l\, F_l = r_r\, F_r$$

Since the angle ϕ between lever beam and weight force is the same on both sides, the equilibrium exists for **any orientation** of the lever beam.

2.3 Concluding Remarks

No dividing of vectors!

- **Dividing** of vectors is **not permitted!**

Vector product only in 3-d

- All the vector operations **with the exception of the vector product** can be transferred to spaces of other dimensions (2-d, 4-d, 5-d, etc.). **The vector product is only defined in three dimensions.**

3 Elementary Functions, Complex Numbers, Power Series Expansion of Functions

3.1 What is a Function?

Independent variable(s) and function value

A function $f: x \to f(x) = y$ is defined as a **unique assignment** of a dependent variable or function value y to an independent variable x by a calculation rule. This can be generalized:

Uniqueness!

A quantity y is a function of one (or several) variable(s) x (or $x_1, x_2, \ldots, x_N$), if <u>one</u> unique value of y can be assigned to any value of x (or any combination $x_1, x_2, \ldots, x_N$, respectively).

Remark: The symbols denoting the independent and the dependent variable are arbitrary. Often, but by no means always, the letters x and y are used. Also the formal symbol f for the function can be replaced, e. g., by the dependent variable, $y = y(x)$.

Examples

1. Perimeter p of a circle as a function of the radius r,

   $$p = p(r) = 2\pi r$$

   in general

   $$y = f(x)$$

2. Conversion of the temperature from centigrade (C) to degrees Fahrenheit (F),

   $$F = F(C) = 32 + \frac{9}{5}C$$

   in general

   $$y = f(x)$$

3. Pressure p of an enclosed quantity of gas of 1 mol as a function of the available volume V and the absolute temperature T according to the ideal gas law,

   $$p = p(V,T) = R\frac{T}{V}$$

   in general

   $$y = f(x_1, x_2)$$

   R is the universal gas constant, a natural constant, hence not a variable $[R = 8{,}3143\ \mathrm{J/(K \cdot mol)}]$.

4. Volume of a liquid transported through a thin capillary according to Hagen-Poiseuille's law (which applies to laminar, i. e., not turbulent, flows),

$$V = V(\Delta p, l, r, t) = \frac{\pi}{8\eta}\frac{\Delta p}{l} r^4 t$$

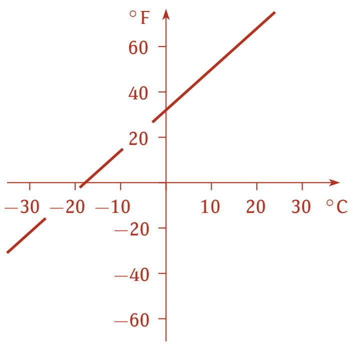
in general

$$y = f(x_1, x_2, x_3, x_4)$$

η denotes the viscosity of the liquid (here assumed constant), Δp the pressure difference between the ends of the capillary, and l and r its length and radius, respectively.

3.2 Representations of a Function

3.2.1 Table



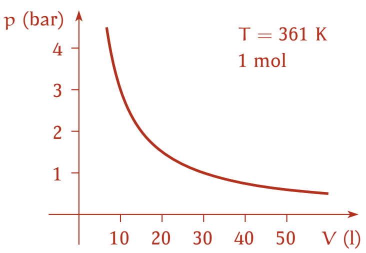
The result of any **experiment** is a table (today usually stored electronically as a data file in the control computer). Examples

1. Temperature conversion from centigrade (°C) to °F,

$$F = F(C) = 32 + \frac{9}{5}C$$

variable C (°C)	5	10	15	20	25	30	35
function value F (°F)	41	50	59	68	77	86	95

	0	−5	−10	−15	−20	−25	−30	−35
	32	23	14	5	−4	−13	−22	−31



2. Pressure of an enclosed quantity of 1 mol of a gas as a function of the volume under isothermal conditions (i. e., for $T = \text{const.}$; here specifically $T = 361$ K). For $T = \text{const.}$, the ideal gas law reduces to **Boyle-Mariotte's law**,

$$p = p(V) = R\frac{T}{V}$$

variable V (l)	6	7.5	10	15	30	60	300
function value p (bar)	5	4	3	2	1	0,5	0,1

Parameter(s)

Remark: Independent variables which are kept constant in a function (such as the temperature in the last example) are dubbed **parameters**. For different values of a parameter, the graph of a function (see below) consists in a **set of curves**.

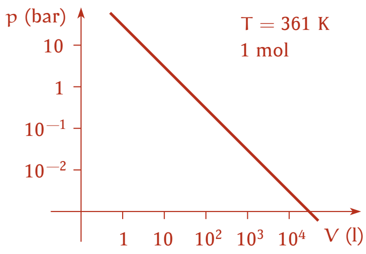
- **Advantage** of the representation as a table: Easy to use; function values are immediately present without calculation
- **disadvantage: Interpolation** required for intermediate values.

3.2.2 Graphical Representation

The dependent variable is plotted versus the independent one in a **2-d diagram**. This yields the **graph** or **curve** of the function. Examples are shown above. In the case of two independent variables, the graph of the function is an **expanse in a 3-d diagram**. The graphical plot has several advantages:

- Continuous representation showing the overall behavior at one glance
- no interpolation required.

Logarithmic representation



If the independent variable and/or the function value varies (vary) over several orders of magnitude, a **semi- or double-logarithmic plot** is often more appropriate than a linear one. Here either one coordinate axis (→ semi-logarithmic plot) or both (→ double-logarithmic plot) is (are) not marked linearly but logarithmically, i. e., powers of 10 are positioned at equal distances on an axis. This has the advantage that the respective variable(s) can be plotted with **constant relative accuracy**. The example on the margin shows Boyle-Mariotte's law for ideal gases, $p = p(V) = RT/V$, for $T = 361$ K in a double-logarithmic (or double-log) plot. A power law such as $p \propto 1/V = V^{-1}$ always yields a straight line in double-log representation, its slope being equal to the exponent of the power law (here, -1).

3.2.3 Analytical Representation

A function being given in analytical representation means that there is an **equation**, i. e., a **mathematical operation** connecting independent variable and function value. Examples are

$y = a + x$	addition
$y = b\,x$	multiplication
$y = x^2$	square
$y = \sqrt{x}$	square root
$y = a^x$	exponential function with arbitrary base $a > 0$
$y = e^x = \exp(x)$	expon. funktion with base $e = 2{,}71828182\ldots$ (Euler's number; see below)
$y = \log_a x$	common logarithm to an arbitrary base $a > 0$
$y = \lg x$	logarithm to the base 10
$y = \ln x$	natural logarithm (base e)
$y = \sin x$ etc.	sine (and other trigonometric functions)
$y = \arcsin x$ etc.	arc sine (and other arc functions)

Analytical function equations usually contain combinations of several elementary functions, such as

$$y = a\,\sin(bx)\,\exp(-cx) + d$$

$$y = \frac{\sin x}{x} + x^3$$

The analytical representation of a function is obtained from a **theory**.

3.2.4 Symbolic Representation

The symbolic representation can be used, if the analytical form of a function is either unknown or unfavorable (e. g., if the equation connecting variable and function value is very complicated). It has already been used,

$y = f(x)$

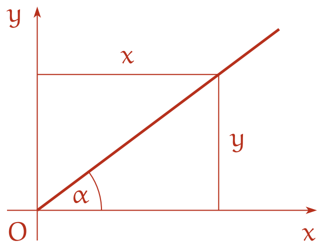
$$y = f(x)$$
$$y = y(x)$$
$$p = p(V)$$
$$F = F(x)$$

or, in the case of more than one variable,

$$p = p(V,T)$$
$$y = y(x_1,x_2,\ldots,x_n)$$
$$g = g(u_1,u_2,\ldots,u_n)$$

3.3 Some Elementary Functions

3.3.1 Linear Function



1. Special case

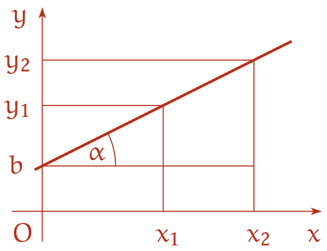
$$y = ax \qquad \text{with } a = \text{const.} \tag{3.1}$$

y is proportional to x. a is the proportionality factor; it determines the **slope of the corresponding straight line**,

$$a = \frac{y}{x} = \tan\alpha \tag{3.2}$$

The slope of a straight is **constant**, i. e., the same for all pairs of values x and y. In this special case the straight runs through the **origin**.

2. General case



$$y = ax + b \qquad \text{with } a,b = \text{const.} \tag{3.3}$$

Here we have

$$a = \frac{y-b}{x} = \tan\alpha \qquad \text{or} \tag{3.4}$$

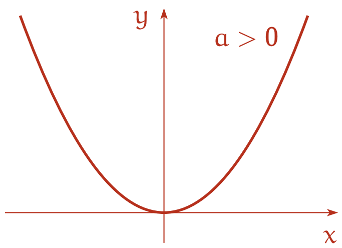
$$a = \frac{y_2 - y_1}{x_2 - x_1} = \tan\alpha \tag{3.5}$$

for all pairs of values (x_1,y_1), (x_2,y_2).

Note

$a > 0$	the straight points upward
$a = 0$	the straight runs horizontally
$a < 0$	the straight points downward.

3.3.2 Square Function; Parabola



1. Special case

$$y = ax^2 \qquad \text{with } a = \text{const.} \tag{3.6}$$

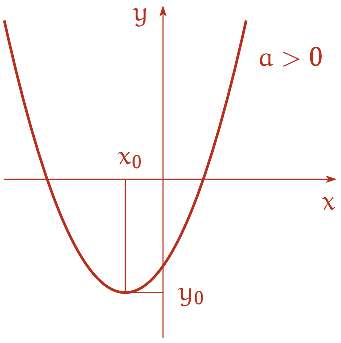
describes a **parabola with its apex at the origin**. The prefactor a determines its steepness.

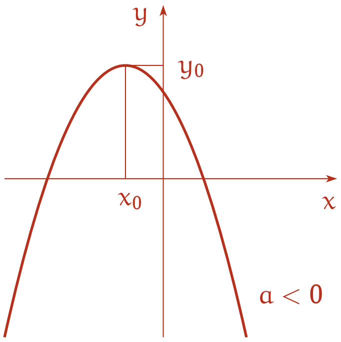
2. General case

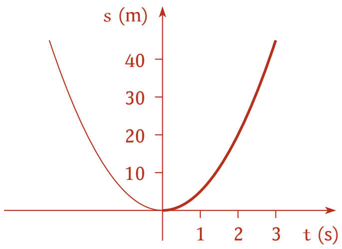
$$y = ax^2 + bx + c \qquad \text{with } a,b,c = \text{const.} \tag{3.7}$$

With the **square complement**, we can always cast this function in the form

$$y - y_0 = a\,(x - x_0)^2 \tag{3.8}$$







with

$$x_0 = -\frac{b}{2a}; \qquad y_0 = c - \frac{b^2}{4a} \tag{3.9}$$

Hence, it describes a **parabola with its apex at** $(\mathbf{x_0}; \mathbf{y_0})$.

Note

$a > 0$	the parabola opens upward
$a = 0$	the function graph is a straight line
$a < 0$	the parabola opens downward

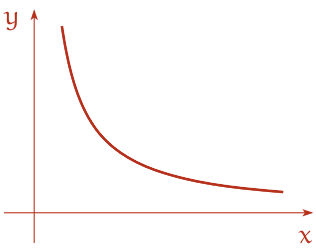
3. Example: **Free fall**

A free falling mass subject to gravity (disregarding its air resistance) moves according to the equation

$$s(t) = \frac{1}{2} g t^2$$

where $g = 9{,}81\ \mathrm{m/s^2}$ denotes the gravitational acceleration (acceleration of free fall). Inserting the time t in seconds after dropping the mass yields the distance $s(t)$ in meters covered by it. The above expression can be calculated for negative times as well; meaningful in terms of physics, however, is only the range $t \geqslant 0$. **Hence, when evaluating mathematical expressions, always keep their real-world meaning in mind!**

3.3.3 Hyperbola



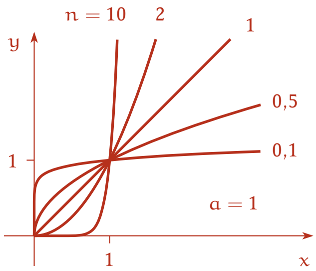
A hyperbola is described by the equation

$$y = \frac{a}{x} = a x^{-1} \tag{3.10}$$

y is **inversely proportional** to x. An example is Boyle-Mariotte's isothermal law for ideal gases mentioned above

$$p(V) = R\,\frac{T}{V}$$

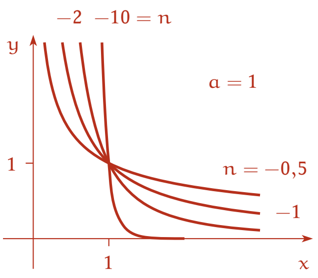
3.3.4 General Power Law



All the functions discussed so far

$$y = ax$$
$$y = ax^2$$
$$y = \frac{a}{x} = ax^{-1}$$

are special cases of the general power law



$$y = ax^n \tag{3.11}$$

with a and n being arbitrary constants.

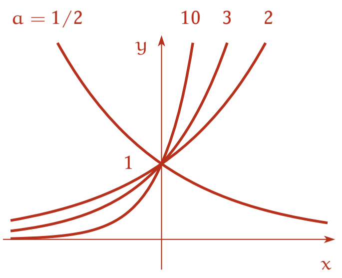
For negative exponents we can write

$$ax^{-n} = \frac{a}{x^n} \tag{3.12}$$

The graphs plotted on the margin belong to the prefactor $a = 1$. All of them run through the point $(1; 1)$.

3.3.5 Exponential Function

In an exponential function, the variable x is in the exponent of an arbitrary **positive** number, the **base** $\boldsymbol{a}$



$$y = a^x \qquad \text{with} \quad a > 0 \tag{3.13}$$

A negative sign in the exponent yields

$$a^{-x} = \frac{1}{a^x} = \left(\frac{1}{a}\right)^x \tag{3.14}$$

i. e., replacing the base with its reciprocal corresponds to mirroring the function graph on the y axis.

- Calculation rules for exponential functions,

$$a^{x_1} \cdot a^{x_2} = a^{(x_1+x_2)} \tag{3.15}$$
$$\frac{a^{x_1}}{a^{x_2}} = a^{(x_1-x_2)} \tag{3.16}$$
$$(a^{x_1})^{x_2} = a^{(x_1 \cdot x_2)} \tag{3.17}$$
$$a^0 = 1 \tag{3.18}$$
$$a^1 = a \tag{3.19}$$

Consequence: All the graphs of simple exponential functions (without a pre-factor) run through the point $(0; 1)$.

- Of particular importance is the exponential function which has **Euler's number** $\boldsymbol{e}$ as base

Euler's number 2.718 ...

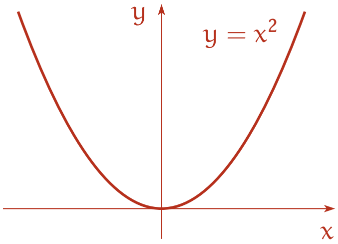
$$y = e^x = \exp(x) \qquad \text{mit}$$
$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 2{,}71828182846\ldots \tag{3.20}$$

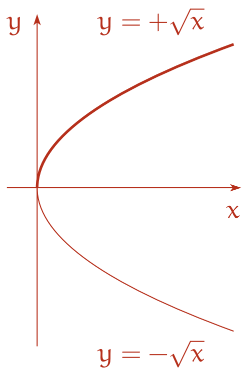
The reasons will become clear in Chapters 4, 5, and 6.

3.3.6 Inverse Function

- We start with an arbitrary function $y = f(x)$ which has x as independent variable. Now we can ask: Is it also possible to express x as a function of y, and how is the functional relation?
- Therefore we solve for x,

$$x = F(y) \tag{3.21}$$





- The names of the variables are unimportant; hence, we exchange x and y in Eq. (3.21), dubbing the independent variable x again,

$$y = F(x) \tag{3.22}$$

Then F is called the **inverse function** to f.

- When calculating an inverse function, keep in mind, however, that the assignment $F\colon x \to F(x)$ must be **unique**, as is required for any function. Hence, if the uniqueness is not fulfilled, part of the co-domain must be excluded.

- Example

$$y = f(x) = x^2$$

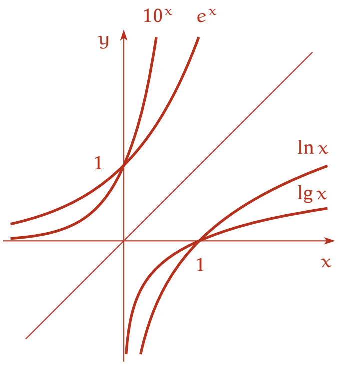
To every x, one unique y is assigned, but there are two values of x to every y, viz. $x_1 = +\sqrt{y}$ and $x_2 = -\sqrt{y}$. After exchanging x and y, we must restrict the co-domain to **one of the two branches of the horizontal parabola**. Usually, the **positive** branch is chosen.

- The graphs of the original function f and its inverse F are transformed into each other by **mirroring on the bisectrix of the first and third quadrant**.

- Other examples

original function	$y = ax + b$	$y = x^n$	$y = a^x$
inverse function	$y = (x - b)/a$	$y = \sqrt[n]{x}$	$y = \log_a x$

3.3.7 Logarithm



The logarithm function $F(x) = \log_a x$ is the **inverse of the exponential function** $f(x) = a^x$ to the base $a > 0$. This means, the logarithm $\log_a x$ is the **exponent required for a to yield x**. Of particular importance are the logarithm to the base 10,

$$y = \log_{10} x = \lg x \tag{3.23}$$

and the logarithm to the base e,

$$y = \log_e x = \ln x \qquad \text{(natural logarithm)} \tag{3.24}$$

The special meaning and importance of the natural logarithm $y = \ln x$ will also become clear in Chapters 4, 5, and 6.

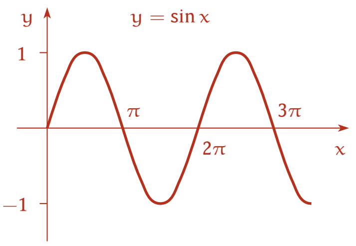
- Some numerical examples of the logrithm $\lg x$ to the base 10

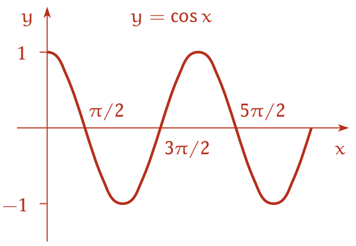
$$\begin{array}{lcl}
10^{-1} = 0{,}1 & \leftrightarrow & -1 = \lg 0{,}1 \\
10^{0} = 1 & \leftrightarrow & 0 = \lg 1 \\
10^{1} = 10 & \leftrightarrow & 1 = \lg 10 \\
10^{2} = 100 & \leftrightarrow & 2 = \lg 100 \\
10^{0{,}5} = 3{,}1623 & \leftrightarrow & 0{,}5 = \lg 3{,}1623 \\
10^{1{,}5} = 31{,}623 & \leftrightarrow & 1{,}5 = \lg 31{,}623
\end{array}$$

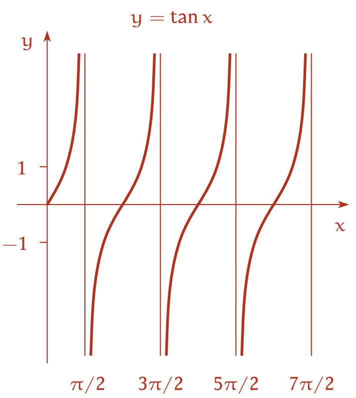
- General calculation rules for logarithms (valid for any base a)

$$\log_a(x_1 \cdot x_2) = \log_a x_1 + \log_a x_2 \tag{3.25}$$
$$\log_a\left(\frac{x_1}{x_2}\right) = \log_a x_1 - \log_a x_2 \tag{3.26}$$
$$\log_a\left(x_1^{x_2}\right) = x_2 \cdot \log_a x_1 \tag{3.27}$$
$$\log_a 1 = 0 \tag{3.28}$$
$$\log_a a = 1 \tag{3.29}$$
$$\log_a x = \frac{1}{\ln a} \cdot \ln x \tag{3.30}$$
$$\log_a x = \frac{1}{\lg a} \cdot \lg x \tag{3.31}$$

3.3.8 Trigonometric functions







The four basic trigonometric functions sine ($\sin x$), cosine ($\cos x$), tangent ($\tan x$), and cotangent ($\cot x$) have already been introduced in **Section 1.1.3 and depicted on the unit circle**. They are periodic with period 2π (sine and cosine) or π (tangent and cotangent), respectively.

- The following important calculation rules apply

$$\sin^2 x + \cos^2 x = 1 \qquad \text{(from Pythagoras's law)} \tag{3.32}$$
$$\tan x = \frac{\sin x}{\cos x} \tag{3.33}$$
$$\cot x = \frac{1}{\tan x} \tag{3.34}$$
$$1 + \tan^2 x = \frac{1}{\cos^2 x} \tag{3.35}$$
$$1 + \cot^2 x = \frac{1}{\sin^2 x} \tag{3.36}$$

- Periodicities

$$\sin(x + k \cdot 2\pi) = \sin x \tag{3.37}$$
$$\cos(x + k \cdot 2\pi) = \cos x \tag{3.38}$$
$$\tan(x + k \cdot \pi) = \tan x \tag{3.39}$$
$$\cot(x + k \cdot \pi) = \cot x \tag{3.40}$$

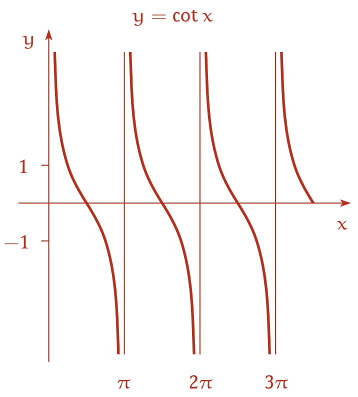
  k can be any whole number including zero.

- Small angles: For small arguments x ($|x| \ll 1$ in radians) we can approximate

$$\sin x \approx \tan x \approx x \tag{3.41}$$
$$\cos x \approx 1 - \frac{1}{2}x^2 \tag{3.42}$$

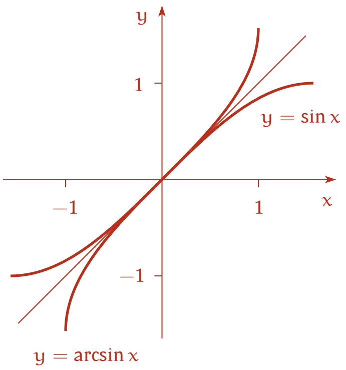
  The proof will be given in Section 3.5.5.

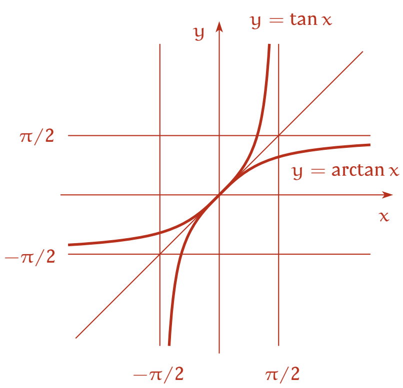
- Addition theorems: Numerous other relations of the trigonometric functions—the functions of sums, differences, multiples, or fractions of arguments, or the expression of one trigonometric function by another—are comprised by the **addition theorems**. They can be derived with the **complex exponential function**, as will be demonstrated in Section 3.4.8. You find them



in the standard formularies of mathematics (e. g., Bronstein-Semendjajew, *Taschenbuch der Mathematik*, publisher Harri Deutsch).

3.3.9 Inverse Trigonometric Functions (Arc Functions)





Arc functions and trigonometric functions are inverse to each other,

$$\begin{array}{lcl} y = \arcsin x & \leftrightarrow & y = \sin x \\ y = \arccos x & \leftrightarrow & y = \cos x \\ y = \arctan x & \leftrightarrow & y = \tan x \\ y = \operatorname{arccot} x & \leftrightarrow & y = \cot x \end{array}$$

The arc functions are sometimes written $y = \sin^{-1} x = \arcsin x$ etc. It is important to keep in mind that in conjunction with the trigonometric functions, the exponent -1 does **not** denote the reciprocal value but the inverse function!

Due to the periodicities of the trigonometric functions, the values of the arc functions would not be unique. Hence, their co-domains are restricted to the following **principal values**

$$-\frac{\pi}{2} \leqslant \arcsin x \leqslant +\frac{\pi}{2} \tag{3.43}$$

$$0 \leqslant \arccos x \leqslant \pi \tag{3.44}$$

$$-\frac{\pi}{2} \leqslant \arctan x \leqslant +\frac{\pi}{2} \tag{3.45}$$

$$0 \leqslant \operatorname{arccot} x \leqslant \pi \tag{3.46}$$

3.4 Complex Numbers

3.4.1 Motivation: Quadratic Equation

The quadratic equation reads

$$ax^2 + bx + c = 0 \tag{3.47}$$

Its general solution can be derived with the square complement and comprises the two values

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.48}$$

Obviously, in the realm of real numbers there is no solution in the case $4ac > b^2$, since the square-root of a negative number is not defined. Consequence: One introduces a new, extended system of numbers, in which the quadratic equation always has solutions. The real numbers should be a sub-set of this new system and, to avoid contradictions, the usual calculation rules should remain valid[1].

3.4.2 Imaginary Unit

Symbolically, we introduce a number i ("imaginary unit"), which, by definition, has the square −1,

$i = \sqrt{-1}$

$$i^2 = -1; \qquad \text{so,} \quad i = \sqrt{-1} \tag{3.49}$$

All the numbers which are the product of a real number and i, are dubbed **imaginary numbers**, e. g.,

$$\sqrt{-9} = \sqrt{-1} \cdot \sqrt{9} = 3i$$

All the numbers which are the sum of a real and an imaginary component, are then dubbed **complex numbers**,

$$z = a + bi; \qquad a, b \text{ real} \tag{3.50}$$

a is the **real part**, b the **imaginary part** of z. Note: The factor i does **not** belong to the the imaginary part. **Real and imaginary part are real numbers!**

3.4.3 The Basic Arithmetic Operations in the Complex World

Since the calculation rules of the real numbers apply, the basic arithmetic operations can easily be performed,

- addition, subtraction

$$\begin{aligned} z_1 \pm z_2 &= (a_1 + ib_1) \pm (a_2 + ib_2) \\ &= \boxed{(a_1 \pm a_2) + i \cdot (b_1 \pm b_2)} \end{aligned} \tag{3.51}$$

- multiplication

$$\begin{aligned} z_1 \cdot z_2 &= (a_1 + ib_1) \cdot (a_2 + ib_2) \\ &= a_1 a_2 + i(a_1 b_2 + a_2 b_1) + i^2 b_1 b_2 \\ &= \boxed{(a_1 a_2 - b_1 b_2) + i \cdot (a_1 b_2 + a_2 b_1)} \end{aligned} \tag{3.52}$$

- division

$$\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} \tag{3.53}$$

To calculate real and imaginary part of the quotient z_1/z_2, use the following **trick**: Enlarge the fraction with $z^* = a_2 - ib_2$. This renders the denominator real,

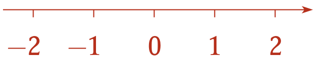
$$\begin{aligned} \frac{z_1}{z_2} &= \frac{(a_1 + ib_1) \cdot (a_2 - ib_2)}{(a_2 + ib_2) \cdot (a_2 - ib_2)} \\ &= \frac{(a_1 a_2 + b_1 b_2) + i(a_2 b_1 - a_1 b_2)}{a_2^2 + b_2^2} \\ &= \boxed{\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i \cdot \frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}} \end{aligned} \tag{3.54}$$

[1] There is one exception, however; see Section 3.4.10.

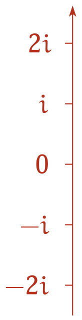
Two complex numbers are equal, if their real and their imaginary parts are equal,

$$\begin{aligned} z_1 = z_2 \quad &\leftrightarrow \quad a_1 = a_2 \quad \text{und} \quad b_1 = b_2 \\ z = 0 \quad &\leftrightarrow \quad a = b = 0. \end{aligned}$$

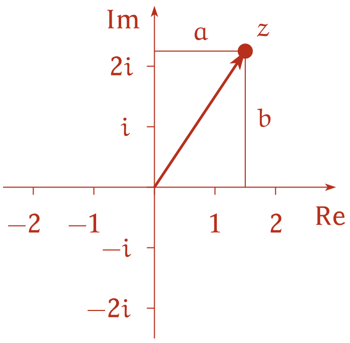
3.4.4 Geometrical Representation: The Complex Plane



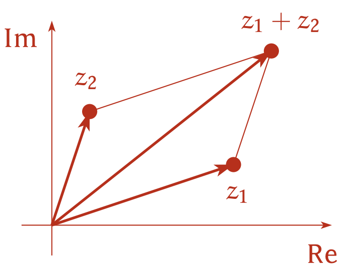
- The real numbers are close lying points on the **real number line**. Usually it is plotted **horizontally**.



- In a similar way, the purely imaginary numbers can be arranged on an **imaginary number line**. It is plotted **vertically**.

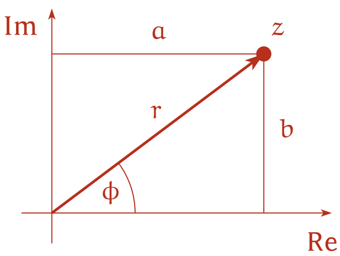


- The two lines can be plotted in a single diagram. They must intersect at the point zero, the only number they have in common. Then they are the axes of a **two-dimensional vector space**, the **complex plane**. Any point or position vector in this plane corresponds to a complex number. Its real and imaginary part are obtained as the projections onto the two axis, as usual.



- Addition and subtraction of two complex numbers correspond to addition and subtraction of the corresponding vectors in the complex plane, respectively (cf. Section 2.2). On the other hand: **Multiplication and division of complex numbers cannot be reconciled with operations in a usual 2-d vector space.** The complex multiplication has nothing to do with the scalar product (which yields a scalar, not a vector) or the vector product (which is only defined in three dimensions). Dividing two vectors is not permitted anyway.

3.4.5 Representation in Polar Coordinates



As we have learned in Section 1.2, points in a plane cannot only be defined with Cartesion coordinates, but with **polar coordinates** as well. This is also valid (and very useful!) for complex numbers,

$$z = a + bi \tag{3.55}$$
$$= r \cdot \cos\phi + r \cdot \sin\phi \cdot i \tag{3.56}$$
$$= r \cdot (\cos\phi + i\,\sin\phi) \tag{3.57}$$
$$= r \cdot e^{i\phi} \tag{3.58}$$

We call

Absolute value and phase

$$r = \sqrt{a^2 + b^2} \qquad \text{the **absolute value** of } z \qquad (3.59)$$

$$\phi = \arctan\left(\frac{b}{a}\right) \qquad \text{the **phase**[2] of } z \qquad (3.60)$$

The relation

$$\cos\phi + i\sin\phi = e^{i\phi} \qquad (3.61)$$

→ Section 3.5.5

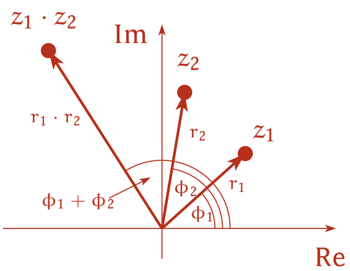
will be proven in Section 3.5.5.

Note: Equation (3.61) has the consequence

$$\left|e^{i\phi}\right| = 1 \qquad (\text{for } \phi \text{ real}) \qquad (3.62)$$

i. e., all the number $e^{i\phi}$ are located on the **unit circle**.

With polar coordinates it is easy to demonstrate the meaning of multiplication, division, and the calculation of powers and roots of complex numbers,



- multiplication

$$z_1 \cdot z_2 = r_1\, e^{i\phi_1} \cdot r_2\, e^{i\phi_2}$$
$$= \boxed{r_1 \cdot r_2 \cdot e^{i(\phi_1+\phi_2)}} \qquad (3.63)$$

(multiply absolute values, add phases)

- division

$$\frac{z_1}{z_2} = \frac{r_1\, e^{i\phi_1}}{r_2\, e^{i\phi_2}}$$
$$= \boxed{\frac{r_1}{r_2} \cdot e^{i(\phi_1-\phi_2)}} \qquad (3.64)$$

(divide absolute values, subtract phases)

- powers

$$z^n = \left(r\, e^{i\phi}\right)^n$$
$$= \boxed{r^n \cdot e^{in\phi}} \qquad (3.65)$$

(raise the absolute value to the power, multiply the phase with the exponent)

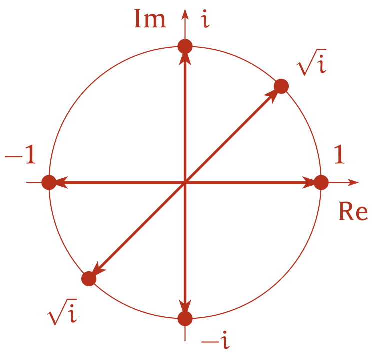
- roots

$$\sqrt[n]{z} = \left(r\, e^{i\phi}\right)^{1/n}$$
$$= \boxed{\sqrt[n]{r} \cdot e^{i\phi/n}} \qquad (3.66)$$

(calculate the root of the absolute value, divide the phase by the root index; but see also the following Section 3.4.6)

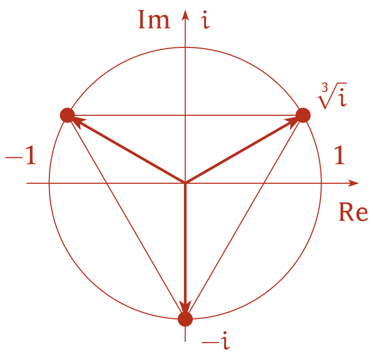
[2] See the footnote on p. 8.

3.4.6 Powers and Roots of i



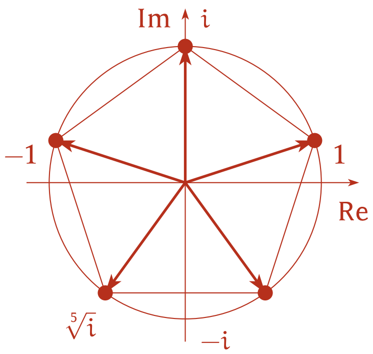
- Since $|i| = 1$, all the powers and roots of i are located on the **unit circle** and can be written as $e^{i\phi}$. Hence, we need to discuss only their phases ϕ. i has the phase $\pi/2$; hence

$$i = e^{i\pi/2} \qquad \text{phase } \pi/2 \tag{3.67}$$
$$i^2 = e^{i\pi} = -1 \qquad \text{phase } \pi \tag{3.68}$$
$$i^3 = e^{i3\pi/2} = -i \qquad \text{phase } 3\pi/2 \tag{3.69}$$
$$i^4 = e^{i2\pi} = 1 \qquad \text{phase } 2\pi \tag{3.70}$$
$$i^5 = e^{i5\pi/2} = i \qquad \text{phase } 5\pi/2 \text{ (or } \pi/2) \tag{3.71}$$



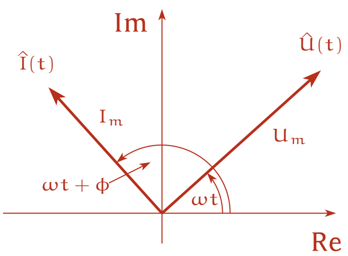
- i has two square-roots with the phases $\pi/4$ and $5\pi/4$, respectively,

$$\left(\sqrt{i}\right)_1 = e^{i\pi/4} = \frac{1+i}{\sqrt{2}} \qquad \text{phase } \pi/4 \tag{3.72}$$
$$\left(\sqrt{i}\right)_2 = e^{i5\pi/4} = \frac{-1-i}{\sqrt{2}} \qquad \text{phase } 5\pi/4 \tag{3.73}$$



- Note: Any complex number (except zero) has two square-roots, three cubic roots, four fourth roots, and—in general—n roots with index n. These are located on a regular polygon with n corners, centered around zero. The reason is that the phase of a complex number is not unique but can only be determined modulo 2π. The n^{th} roots of a number with phase ϕ have the phases $(\phi + 2k\pi)/n$ mit $k = 0, 1, 2, \ldots, n-1$. The plots on the margin show the three cubic roots and the five 5^{th} roots of i as an example.

3.4.7 Phasor Diagrams



The relation between voltage and current in **electronic elements in ac circuits**, e. g., solenoids and capacitors, is often illustrated with so-called **phasor diagrams**. Voltage and current are plotted as vectors (or "phasors") in a plane with the absolute values U_m and I_m corresponding to maximum voltage and current, respectively. Their starting points lie in the coordinate origin; the angle between them represents the phase shift ϕ between voltage and current. The two vectors rotate together in the plane with the angular frequency ω of the ac voltage. The sketch shows the phasor diagram of a **capacitor** with $\phi = \pi/2$ as an example.

- The phasor diagrams can now easily be understood as representations of voltage and current in the **complex plane**,

$$\hat{U}(t) = U_m \cdot \exp(i\omega t) \qquad \text{voltage} \tag{3.74}$$
$$\hat{I}(t) = I_m \cdot \exp[i(\omega t + \phi)] \qquad \text{current} \tag{3.75}$$

- The physical values of voltage and current at any instant of time are the **real parts** of the complex numbers, i. e., their projections onto the real axis,

$$U(t) = U_m \cdot \cos(\omega t) \tag{3.76}$$
$$I(t) = I_m \cdot \cos(\omega t + \phi) \tag{3.77}$$

- The quotient $\hat{Z} = \hat{U}_m/\hat{I}_m = (U_m/I_m) \cdot e^{-i\phi}$ is the **complex ac impedance** of the corresponding element. It contains both the absolute value of the impedance or resistance ($|\hat{Z}| = U_m/I_m$) and the phase shift ϕ.

3.4.8 Addition Theorems

The addition theorems describe relations between the trigonometric functions. They can be easily derived with the aid of the **complex exponential function**. As an example, let us express the sine and cosine value of a sum or difference of two angles by the functions of the single arguments,

$$\sin(\alpha \pm \beta) = ? \tag{3.78}$$
$$\cos(\alpha \pm \beta) = ? \tag{3.79}$$

The complex exponential function is expanded

$$e^{i(\alpha \pm \beta)} = \cos(\alpha \pm \beta) + i \sin(\alpha \pm \beta) \tag{3.80}$$

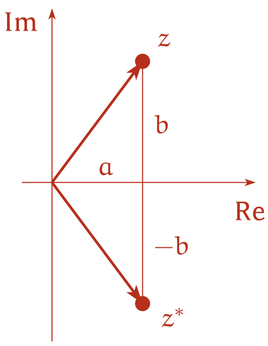
and, on the other hand,

$$\begin{aligned} e^{i(\alpha \pm \beta)} &= e^{i\alpha}\, e^{\pm i\beta} \\ &= (\cos\alpha + i \sin\alpha) \cdot (\cos\beta \pm i \sin\beta) \\ &= \cos\alpha \cos\beta \mp \sin\alpha \sin\beta + i(\sin\alpha \cos\beta \pm \cos\alpha \sin\beta) \end{aligned} \tag{3.81}$$

The **comparison of the imaginary and real parts** of Eqs. (3.80) and (3.81) yields the addition theorems,

$$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta \tag{3.82}$$
$$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta \tag{3.83}$$

3.4.9 Conjugate Complex Number



We start with an arbitrary complex number,

$$z = a + bi \tag{3.84}$$

Its **conjugate complex number**

$$z^* = a - bi \tag{3.85}$$

is defined as the number with the same real part and the **negative imaginary part** of z ($b \to -b$). Geometrically, this corresponds to **mirroring on the real axis**.

- Note: We have

$$z \cdot z^* = (a + bi) \cdot (a - bi) = a^2 + b^2 = r^2 \tag{3.86}$$

(real). We have made use of this trick already in performing the complex division [Gl. (3.54)]. Then we can calculate the absolute value r of a complex number as

$r = \sqrt{z \cdot z^*}$

$$r = \sqrt{z \cdot z^*} \tag{3.87}$$

- Furthermore we have

$$\mathrm{Re}(z) = a = \frac{1}{2}(z + z^*) \tag{3.88}$$

$$\mathrm{Im}(z) = b = \frac{1}{2i}(z - z^*) \tag{3.89}$$

3.4.10 Comparing Operations

$<, >$ etc. **not permitted!**

Attention: Comparing operations ($<, >, \leqslant, \geqslant$) **cannot** be used in the realm of complex numbers, since they would give rise to contradictions. Hence, it is not possible to decide, e. g., whether $i < 1$ or $i > 1$ holds. Only real quantities such as absolute values, real and imaginary parts can be compared.

3.4.11 So Why Complex Numbers?

We have introduced the complex numbers in Sections 3.4.1 and 3.4.2 for the—a priori formal—purpose of being able to solve the quadratic equation in all cases, i. e., to calculate the square-root of a negative number. With the polar-coordinate representation in the complex plane it is now straightforward to see, why and how this works.

Positive real numbers have the phase 0 or 2π, so their square-roots have 0 or π and are located on the real axis as well. The square-root of a negative number (with phase π or 3π), on the other hand, has $\pi/2$ or $3\pi/2$ and sits on the imaginary axis.

The phase is, in general, an important parameter of complex numbers, since it defines the phase or phase difference of phenomena in physics—in particular of those with periodic behavior. In Section 3.4.7 we have discussed the phase shift between current and voltage in ac circuits as an example. Another important example is the phase shift between momentary elongation and excitation in driven oscillations. It depends on the excitation frequency and can be calculated with the complex exponential function in an elegant and easy way.

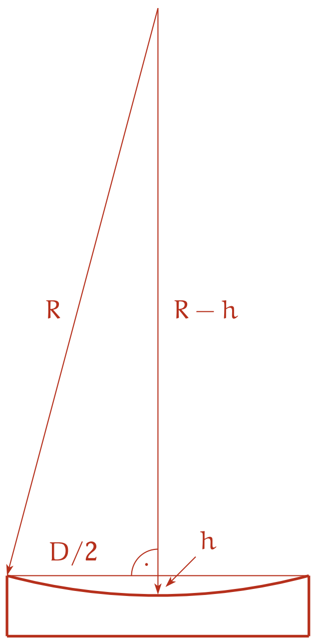
3.5 Power Series Expansion of Functions

3.5.1 Motivation

An amateur astronomer plans to grind the primary mirror for her new back-yard telescope. Diameter D and radius of curvature R of the mirror are given. She calculates the maximum depth, to which the spherical surface must be ground into the glass blank (the so-called sagitta h).

- Pythagoras's law holds that

$$(R - h)^2 + \left(\frac{D}{2}\right)^2 = R^2 \tag{3.90}$$



Solving for the sagitta h yields

$$h = R \cdot \left(1 - \sqrt{1 - \frac{D^2}{4R^2}}\right) \tag{3.91}$$

or

$$\frac{h}{R} = 1 - \sqrt{1 - \frac{a^2}{4}} \qquad \text{with } a = \frac{D}{R} \tag{3.92}$$

The dimensionless quantity a describes the relation between diameter and radius of curvature of the mirror.

- In small amateur telescopes, mainly primary mirrors of long focal length are used, i. e., $D \ll R$ or $a \ll 1$. In this limit, Eq. (3.92) can be well approximated by the following simpler expression

$$\frac{h}{R} \approx \frac{a^2}{8} \tag{3.93}$$

The approximation becomes better for smaller[3] a,

a	0.1	0.3	0.7	1.0	1.5
h/R (exact)	0.00125	0.01131	0.0633	0.134	0.339
$a^2/8$	0.00125	0.01125	0.0613	0.125	0.281

- The approximation $a^2/8$ is the first term of the **power series expansion of the exact formula** [Gl. (3.92)]. It can be improved by adding more terms with higher powers of a.

3.5.2 Series and Power Series

- In mathematics, a **series** is a sum of terms, e. g.,

$$s = a_0 + a_1 + a_2 + \cdots + a_n + \cdots \tag{3.94}$$

or, using the sum sign,

$$s = \sum_{n=0}^{\infty} a_n \tag{3.95}$$

The series in this example contains an infinite number of terms.

- Steady functions $f(x)$ can usually be written as a **power series** (also called **Taylor series**) of their independent variable x,

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \tag{3.96}$$

or, with the sum sign,

Expansion around $x_0 = 0$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{3.97}$$

The definition of the term "steadiness" and further formal prerequisites for the applicability of the power series expansion are given in the textbooks of mathematics.

[3] The primaries used in small amateur telescopes usually have a values between 0.06 and 0.3.

- In Eqs. (3.96) and (3.97), the function is expanded around the point $x_0 = 0$. The expansion can be performed around other points as well. This is even mandatory, if the function is not defined at $x_0 = 0$ (or in the vicinity of $x_0 = 0$). For example, the logarithm and the square-root function are usually expanded around $x_0 = 1$ as in Eq. (3.92).[4] Eqs. (3.96) and (3.97) must then be generalized to

$$f(x_0 + x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \tag{3.98}$$

or

Expansion around any x_0

$$f(x_0 + x) = \sum_{n=0}^{\infty} a_n x^n \tag{3.99}$$

Defining $\xi = x_0 + x$, we can write alternatively

$$f(\xi) = a_0 + a_1 (\xi - x_0) + a_2 (\xi - x_0)^2 + \cdots + a_n (\xi - x_0)^n + \cdots \tag{3.100}$$

or

$$f(\xi) = \sum_{n=0}^{\infty} a_n (\xi - x_0)^n \tag{3.101}$$

- The zero-order term is simply obtained as $a_0 = f(x_0)$. Calculating the coefficients of the higher orders a_n with $n \geqslant 1$ requires differential calculus; see Section 4.6.2.

- In practical applications, the number of terms to be included in a power series expansion depends on the required precision. If a theory is to be compared with experimental data points, for instance, the power series can usually be truncated when the remaining error is smaller than the experimental error bars. In many cases the term of lowest order (or, perhaps, the next one) is already sufficient.

3.5.3 Factorial

$n!$ ("n-factorial")

For the following sections we need the factorial ($n!$). It is defined for the natural numbers n including zero:

$$0! = 1 \tag{3.102}$$
$$1! = 1 \tag{3.103}$$
$$2! = 1 \cdot 2 = 2 \tag{3.104}$$
$$3! = 1 \cdot 2 \cdot 3 = 6 \tag{3.105}$$
$$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24 \tag{3.106}$$
$$\vdots$$
$$n! = n \cdot (n-1)! \qquad \text{for } n \geqslant 1 \tag{3.107}$$

[4] This corresponds to the limit $a \ll 1$ in Eq. (3.92).

3.5.4 Power Series of Some Selected Functions

1. Geometrical series

$$\frac{a}{1-x} = a\,(1+x+x^2+x^3+x^4+\cdots) = a\sum_{n=0}^{\infty} x^n \tag{3.108}$$

(range of convergence, $|x| < 1$)

All the coefficients are equal here.

2. Natural logarithm

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \tag{3.109}$$

(range of convergence, $-1 < x \leqslant +1$)

3. Square-root function

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1\cdot 1}{2\cdot 4}x^2 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}x^3 - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 + \cdots \tag{3.110}$$

(range of convergence, $|x| \leqslant 1$)

Note: In Eq. (3.92) we have $x = -a^2/4$; hence, the approximation of lowest order [Eq. (3.93)] is quadratic in a. Also the higher-order terms contain only even powers of a.

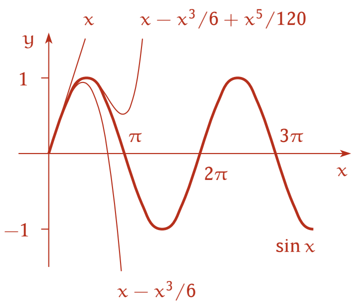
4. Exponential function

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \tag{3.111}$$

$$e^{-x} = 1 - \frac{1}{1!}x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \tag{3.112}$$

(range of convergence, $|x| < \infty$)

5. Sine



$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \tag{3.113}$$

(range of convergence, $|x| < \infty$)

The plot shows the exact sine function and its first three approximations.

6. Cosine

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad (3.114)$$

(range of convergence, $|x| < \infty$)

7. Tangent

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \cdots \quad (3.115)$$

(range of convergence, $|x| < \pi/2$)

$\sin x \approx \tan x \approx x$
$\cos x \approx 1 - \frac{1}{2}x^2$
for $|x| \ll 1$

The last three expansions are the origin of the approximations of the trigonometric functions for small arguments which have been mentioned earlier [Eqs. (3.41) and (3.42)].

For the power series expansions of more functions see the formularies of mathematics (e. g., Bronstein-Semendjajew, *Taschenbuch der Mathematik*, publisher Harri Deutsch).

3.5.5 Euler's Formulas

A look at the power series of the exponential, the sine, and the cosine function shows that they all have similar coefficients.

- To see their relationship in more detail, we start with the power series of the exponential function [Eq. (3.111)] and replace its variable x with ix,

$$e^{ix} = 1 + \frac{1}{1!}ix + \frac{1}{2!}(ix)^2 + \frac{1}{3!}(ix)^3 + \frac{1}{4!}(ix)^4 + \cdots \quad (3.116)$$

- After calculating the powers of i we separate the real and the imaginary terms

$$e^{ix} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$
$$+ i \cdot \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \right) \quad (3.117)$$

and, comparing the result with Eqs. (3.113) und (3.114), we see that

$e^{ix} = \cos x + i \sin x$

$$\boxed{e^{ix} = \cos x + i \sin x} \quad (3.118)$$

This equation has been used in the polar-coordinate representation of complex numbers [Eq. (3.57) and (3.58)].

- In a similar way we can derive

$e^{-ix} = \cos x - i \sin x$

$$\boxed{e^{-ix} = \cos x - i \sin x} \quad (3.119)$$

→ Sections 3.4.7 and 6.3

- The equations (3.118) and (3.119) are called **Euler's formulas**. They demonstrate that, for purely imaginary argument, the exponential function has an **oscillatory, i. e., periodic behavior** and assumes solely values on the unit circle in the complex plane. Hence, it describes **vibration and wave phenomena** in an elegant way. See also Sections 3.4.7 and 6.3.

- Eqs. (3.118) and (3.119) finally yield

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \tag{3.120}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2\,i} = -\frac{i}{2}\left(e^{ix} - e^{-ix}\right) \tag{3.121}$$

4 Differential Calculus

4.1 Basic Ideas

In Chapters 4 and 5 we discuss the **(infinitesimal) calculus** which comprises the two branches **differential and integral calculus**. Infinitesimal calculus means **calculating with indefinitely small numbers**. This is possible and yields meaningful (often finite) results, provided that the rules are obeyed.

Differentiation and integration are "inverse" operations which can be applied to functions. Here we will perform the calculus—apart from a few simple exceptions—only with real functions of one or several real variable(s). The extension to the whole complex plane leads to **complex analysis**, an important field of mathematics.

4.1.1 The Increment

$\Delta x = x_2 - x_1$

Δx, Δy, etc.

- Let us consider an independent variable x. The variation of x from a start value x_1 to a final value x_2 corresponds to an **increment** $\Delta x = x_2 - x_1$. The increment can be positive or negative, depending on the relation $x_2 > x_1$ or $x_2 < x_1$, respectively. The symbol Δ in general denotes (finite) increments of variables or function values.
- For a function $y = f(x)$, an increment in x has the consequence that also the function value y receives an increment, which can again be positive or negative, depending on the x values,

$$y_1 = f(x_1) \quad (4.1)$$

$$y_2 = f(x_2) \quad (4.2)$$

  and, thus,

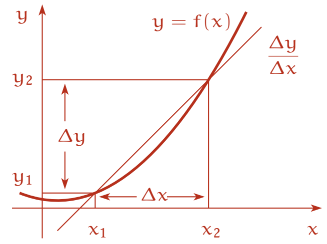
$$\Delta y = y_2 - y_1 = f(x_2) - f(x_1) \quad (4.3)$$

- In the case of the linear function (straight line), Δy depends only on the increment Δx: Δy is **proportional** to Δx. For all other functions, Δy depends on the individual values x_1 and x_2 as well.

4.1.2 The Difference Quotient

- We have again a given function $f\colon x \to f(x) = y$, an increment $\Delta x = x_2 - x_1$ of the variable, and the corresponding increment $\Delta y = y_2 - y_1$ of the function value. For better comparison of the increments we divide Δy by Δx and obtain the **difference quotient**

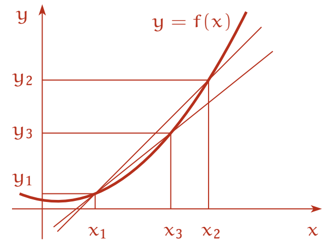
$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad (4.4)$$



- In the case of the **linear function (straight line)** $y = ax + b$, the difference quotient is **constant**, i. e., the same for all pairs $(x_1; x_2)$.

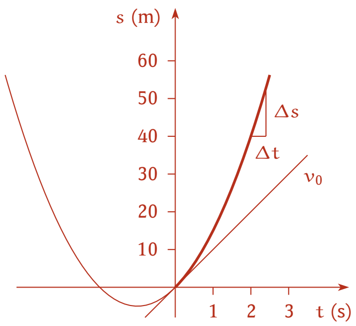
$$\frac{\Delta y}{\Delta x} = a = \text{const.} \tag{4.5}$$

- For **any function**, the difference quotient is the **average relative variation** of the function value in the interval $(x_1; x_2)$.



- The difference quotient is equal to the slope of the **chord (secant)** between the points $(x_1; y_1)$ and $(x_2; y_2)$ on the function graph. Except for the straight, it depends on the width of the interval,

$$\frac{y_3 - y_1}{x_3 - x_1} \neq \frac{y_2 - y_1}{x_2 - x_1} \tag{4.6}$$



- Example from physics: **Free falling mass with non-zero start velocity**. A mass which has been thrown vertically with non-zero start velocity v_0 accelerates under the influence of Earth's gravity. We want to calculate its fall velocity as a function of time.

  The distance traveled up to time t is

$$s(t) = v_0\,t + \frac{1}{2}gt^2$$

  During the time interval Δt the mass covers the additional distance

$$\Delta s = s(t + \Delta t) - s(t)$$

  Hence, $\Delta s/\Delta t$ is the **average velocity** on its way Δs between $s(t)$ and $s(t + \Delta t)$. It is calculated as

$$\begin{aligned}\frac{\Delta s}{\Delta t} &= \frac{1}{\Delta t}\left[\frac{1}{2}g(t+\Delta t)^2 + v_0\,(t + \Delta t) - \frac{1}{2}gt^2 - v_0\,t\right] \\ &= gt + v_0 + \frac{1}{2}g\Delta t\end{aligned}$$

  The shorter the time interval Δt is chosen, the less varies the average velocity from one interval to the next. Finally, in the limit $\Delta t \to 0$ it becomes independent of the interval width and tends to the value $gt + v_0$, the true fall velocity at time t.

4.1.3 Differential Quotient, Derivative

- To determine the relative variation of a function (i. e., the slope of its graph) independent of the interval width, we calculate the **limit of the difference quotient for $\Delta x \to 0$**. We write

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{4.7}$$

  or

$\frac{dy}{dx} = y'(x)$

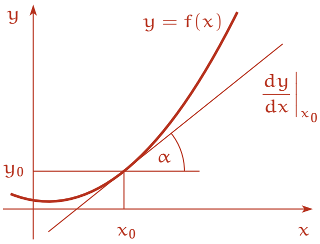
$$\frac{dy}{dx} = y'(x) = f'(x) = \frac{df}{dx} = \frac{d}{dx}f(x) \tag{4.8}$$

- dy/dx is called the **differential quotient** or the **derivative** of the function $y = f(x)$ with respect to x (spoken "dy by dx"). In general, it is again a function of the variable.
- In our example of the free falling mass, the fall velocity at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = gt + v_0$$

It increases linearly with time.

4.1.4 Geometrical Meaning of the Derivative



As the interval width of the independent variable tends to zero ($\Delta x \to 0$), the **secant** becomes the **tangent** which touches the function graph in a single point x_0 and has the same slope there. Hence, the derivative is equal to the **slope of the tangent at the point** x_0 and equal to the tangent function of its angle α relative to the horizontal.

4.1.5 Examples

1. Let $y(t)$ be the **amount of a substance** during a chemical reaction. Then

$$\frac{dy}{dt} = \dot{y}(t)$$

is the **reaction rate**, i. e., the **rate of generation or consumption** of the substance. Note:

$\frac{dy}{dt} = \dot{y}(t)$

temporal derivatives are often denoted $\dot{y}$ rather than y'

2. Let $Q(T)$ be the **heat content** of a sample at the absolute temperature T. Then

$$\frac{dQ}{dT} = C(T)$$

is the **heat capacity** of the sample at that temperature. By the way: The heat capacity of one mole (called **specific heat**) and its temperature dependence are important quantities in thermodynamics and solid-state physics.

3. Let $n(t)$ be the **number of individuals** of a population (e. g., a culture of bacteria) at time t. Then

$$\frac{dn}{dt} = \dot{n}(t)$$

is its **growth rate** (cf. Section 6.3).

4. The **electrical potential** around a point charge Q varies with the distance r from the charge according to

$$\phi(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r}$$

with $\epsilon_0 = 8{,}8542 \cdot 10^{-12}$ As/Vm the permittivity of vacuum, a universal constant. The **electric field strength** $\vec{E}$ generated by the charge is a vector which, for symmetry reasons, points radially away from, or towards, the charge (depending on its sign). The magnitude of the field strength is related to the potential by

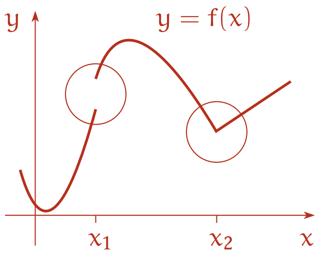
$$E(r) = -\frac{d\phi}{dr}$$

We will calculate $E(r)$ in similar way as the free-fall velocity above,[1]

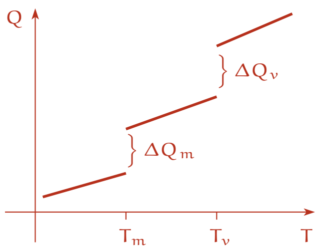
$$\begin{aligned} E(r) &= -\frac{d\phi}{dr} = -\lim_{\Delta r \to 0} \frac{\Delta\phi}{\Delta r} \\ &= -\lim_{\Delta r \to 0} \frac{1}{\Delta r} \frac{Q}{4\pi\epsilon_0} \left[\frac{1}{r+\Delta r} - \frac{1}{r} \right] \\ &= -\frac{Q}{4\pi\epsilon_0} \lim_{\Delta r \to 0} \frac{1}{\Delta r} \frac{r-(r+\Delta r)}{r\cdot(r+\Delta r)} \\ &= \frac{Q}{4\pi\epsilon_0} \lim_{\Delta r \to 0} \frac{1}{r^2 + r\Delta r} \\ &= \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \\ &= E_c(r) \end{aligned}$$

The result is the well-known Coulomb field strength $E_c(r)$ of a point charge, as expected.

4.1.6 Differentiability

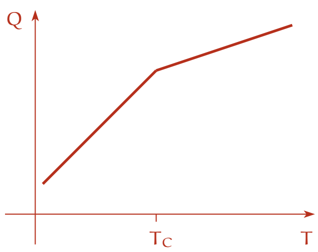


In order for a function $f\colon x \to f(x)$ to be differentiable at point x_0, it must meet two requirements. First, it needs to be **continuous** there, and second, its derivative must **tend to the same value** when approaching x_0 from the left and the right side. This means that its graph must have **no jump and no kink** at x_0. The function sketched on the margin is not differentiable at points x_1 and x_2.



Examples from thermodynamics: Consider again the **heat content** $Q(T)$ of a substance in the vicinity of **phase transitions**.

1. **Phase transitions of first order:** In order to melt or evaporate a substance, we must supply it with heat, i. e., energy, although its temperature does not change. The consumed energy is called **latent heat** (melting heat ΔQ_m and heat of evaporation ΔQ_v, respectively); it is required for the transition between two states of matter with different degrees of order. (Also the variation of the heat content Q with temperature T is, in general, different above and below the phase transition temperature.) Hence, the heat capacity $C(T) = dQ/dT$ is not defined at the melting temperature T_m and the boiling temperature T_v; the function $Q(T)$ is **not differentiable** there.



2. **Phase transitions of second order:** Ferromagnetic metals such as iron have the property of being ferromagnetic only below a certain temperature, the **Curie temperature T_C**. For iron, e. g., it is 770 °C (1418 °F). At temperatures above T_C, the thermal motion prevents the magnetic moments of the atoms

[1] For calculating the components of the vector $\vec{E}$ we need partial derivatives; see Section 4.8.5.

from aligning and the metal becomes paramagnetic. The heat content is continuous at the Curie temperature but its derivative $C(T) = dQ/dT$ changes abruptly. Hence, also in this case $Q(T)$ is **not differentiable** at T_C.

4.2 Derivatives of Elementary Function

4.2.1 Small Look-Up Table

$$y(x) = \text{const.} \qquad y'(x) = 0 \tag{4.9}$$

$$y(x) = x^n \qquad y'(x) = n\,x^{n-1} \quad (n \text{ beliebig}) \tag{4.10}$$

$$y(x) = \sin x \qquad y'(x) = \cos x \tag{4.11}$$

$$y(x) = \cos x \qquad y'(x) = -\sin x \tag{4.12}$$

$$y(x) = \tan x \qquad y'(x) = \frac{1}{\cos^2 x} \tag{4.13}$$

$$y(x) = \cot x \qquad y'(x) = -\frac{1}{\sin^2 x} \tag{4.14}$$

$$y(x) = e^x \qquad y'(x) = e^x \tag{4.15}$$

$$y(x) = a^x \qquad y'(x) = \ln a \cdot a^x \tag{4.16}$$

$$y(x) = \ln x \qquad y'(x) = \frac{1}{x} \tag{4.17}$$

$$y(x) = \log_a x \qquad y'(x) = \frac{1}{\ln a} \cdot \frac{1}{x} \tag{4.18}$$

4.2.2 Examples

$$y(x) = x \qquad y'(x) = 1 \tag{4.19}$$

$$y(x) = x^2 \qquad y'(x) = 2x \tag{4.20}$$

$$y(x) = \frac{1}{x^5} = x^{-5} \qquad y'(x) = -5x^{-6} = -\frac{5}{x^6} \tag{4.21}$$

$$y(x) = \sqrt{x} = x^{1/2} \qquad y'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \tag{4.22}$$

$$y(x) = \frac{1}{\sqrt{x}} = x^{-1/2} \qquad y'(x) = -\frac{1}{2}x^{-3/2} = -\frac{1}{2\sqrt{x^3}} \tag{4.23}$$

$$y(x) = \sqrt[3]{x^5} = x^{5/3} \qquad y'(x) = \frac{5}{3}x^{2/3} = \frac{5}{3}\sqrt[3]{x^2} \tag{4.24}$$

4.2.3 Three Illustrative Proofs

In most cases, mathematical proofs are omitted in the present tutorial. Only three simple examples are given to illustrate the general procedure.

1. **Power Function with Positive Integer Exponent:** $y = x^n$

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[x^n + n\,x^{n-1}\Delta x + \frac{n(n-1)}{2}x^{n-2}(\Delta x)^2 \right. \\ &\qquad \left. + \cdots - x^n\right]\end{aligned}$$

$$= n\,x^{n-1} + \lim_{\Delta x \to 0} \frac{n(n-1)}{2} x^{n-2} \Delta x + \cdots$$
$$= \boxed{n\,x^{n-1}}$$

Expanding the bracket $(x + \Delta x)^n$ yields terms of the form x^{n-k} $(\Delta x)^k$; $k = 0, \ldots, n$ with their pre-factors. Dividing by Δx and subsequently performing the limit $\Delta x \to 0$ cancels all terms with $k \geqslant 2$.

2. **Natural logarithm: $y = \ln x$**

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\ln(x + \Delta x) - \ln x}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{1}{x} \frac{\ln(1 + \Delta x / x)}{\Delta x / x}$$
$$= \frac{1}{x} \lim_{\Delta x \to 0} \left(\frac{\Delta x}{x}\right)^{-1} \left[\frac{\Delta x}{x} - \frac{1}{2}\left(\frac{\Delta x}{x}\right)^2 + \frac{1}{3}\left(\frac{\Delta x}{x}\right)^3 - \frac{1}{4}\left(\frac{\Delta x}{x}\right)^4 + \ldots\right]$$
$$= \boxed{\frac{1}{x}}$$

In the third line we used the power series expansion of the ln function; see Section 3.5.4. The quadratic and all higher orders of the series cancel again in the limit $\Delta x \to 0$.

3. **Exponential function: $y = e^x$**

$$e^x = 1 + \frac{1}{1!} x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \cdots$$
$$\frac{dy}{dx} = 0 + 1 + \frac{2}{2!} x + \frac{3}{3!} x^2 + \frac{4}{4!} x^3 + \frac{5}{5!} x^4 + \cdots$$
$$= 1 + \frac{1}{1!} x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \cdots$$
$$= \boxed{e^x}$$

Differentiation of the power series yields zero for the constant term 1. In an infinite series this does not matter: Every term of the original series is also present in the derivative.

4.3 Rules for Composite Functions

4.3.1 Sum Rule

$$f(x) = g(x) + h(x) \qquad f'(x) = g'(x) + h'(x) \tag{4.25}$$

or

$$y = u + v \qquad y' = u' + v' \tag{4.26}$$

4.3.2 Constant Pre-Factor

$$f(x) = c \cdot g(x) \qquad f'(x) = c \cdot g'(x) \tag{4.27}$$

or

$$y = c \cdot u \qquad y' = c \cdot u' \tag{4.28}$$

4.3.3 Combination of 4.3.1 und 4.3.2

$$f(x) = c_1 \cdot g(x) + c_2 \cdot h(x) \quad f'(x) = c_1 \cdot g'(x) + c_2 \cdot h'(x) \tag{4.29}$$

or

$$y = c_1 \cdot u + c_2 \cdot v \qquad y' = c_1 \cdot u' + c_2 \cdot v' \tag{4.30}$$

This intuitive rule for derivatives has already been applied in the last proof of Section 4.2.3.

4.3.4 Product Rule

$$f(x) = g(x) \cdot h(x) \quad f'(x) = g'(x) \cdot h(x) + g(x) \cdot h'(x) \tag{4.31}$$

or

$$y = u \cdot v \qquad y' = u' \cdot v + u \cdot v' \tag{4.32}$$

similarly for more than two factors,

$$y = u \cdot v \cdot w \qquad y' = u' \cdot v \cdot w + u \cdot v' \cdot w + u \cdot v \cdot w' \tag{4.33}$$

etc.

4.3.5 Quotient Rule

$$f(x) = \frac{g(x)}{h(x)} \qquad f'(x) = \frac{h(x)\, g'(x) - g(x)\, h'(x)}{[h(x)]^2} \tag{4.34}$$

or

$$y = \frac{u}{v} \qquad y' = \frac{v\, u' - u\, v'}{v^2} \tag{4.35}$$

The correct order in the numerator is easy to remember this way: Write the undifferentiated function $h(x)$ squared in the denominator and—unsquared and multiplied with $g'(x)$—as the first term of the numerator.

4.3.6 Derivative of the Reciprocal Function

$$f(x) = \frac{1}{h(x)} \qquad f'(x) = -\frac{h'(x)}{[h(x)]^2} \tag{4.36}$$

or

$$y = \frac{1}{v} \qquad y' = -\frac{v'}{v^2} \tag{4.37}$$

This is a special case of the quotient rule (see above).

4.3.7 Chain Rule

Consider a composed function of the form

$$f(x) = g\,[h(x)] \tag{4.38}$$

or

$$f(x) = g(y) \qquad \text{with} \quad y = h(x) \tag{4.39}$$

Its derivative is

$$f'(x) = \left[\frac{d}{dy}\,g(y)\right] \cdot \left[\frac{d}{dx}\,h(x)\right] = g'(y) \cdot h'(x) \tag{4.40}$$

Don't forget the inner derivative!

Both the outer function $g(y)$ and the inner function $y = h(x)$ are differentiated with respect to their respective arguments, and the results are multiplied.

Example

$$f(x) = \cos^2 x = (\cos x)^2 = y^2 \qquad \text{with} \quad y = \cos x$$

$$f'(x) = 2y \cdot (-\sin x) = -2\sin x\,\cos x = -\sin(2x)$$

The last step $-2\sin x\,\cos x = -\sin(2x)$ is one of the addition theorems for trigonometric functions; it is not related to the derivative and the chain rule.

4.4 Derivative of the Inverse Function

- Let two functions f and F be inverse to each other, i. e.,

  $$y = f(x) \tag{4.41}$$

  and

  $$x = F(y) \tag{4.42}$$

- We differentiate both sides of Eq. (4.42) with respect to x. Left-hand side

  $$\frac{d}{dx}\,x = 1 \tag{4.43}$$

  right-hand side (using the chain rule)

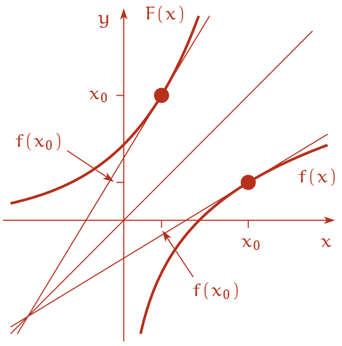
  $$\frac{d}{dx}\,F(y) = \frac{dF}{dy} \cdot \frac{df}{dx} \tag{4.44}$$

- Equating the right-hand sides of Eqs. (4.43) and (4.44) yields

  $$\frac{df}{dx} = \frac{1}{\dfrac{dF}{dy}} \tag{4.45}$$

  or

  $$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}} \tag{4.46}$$



- **The derivative of the inverse function $F(y)$ is reciprocal to the derivative of the original function $f(x)$ and vice versa.** This confirms the previous result that the graphs of the two funcitons are related by mirroring on the bisectrix of the 1st and 3rd quadrant.
- We can—as usual—rename F's variable x, but then we must keep in mind that the derivatives of f and F, which are related by Eq. (4.45), **have to be calculated at different points of the x axis.** The sketch on the margin depicts this. A pair of variables $[x_0; f(x_0)]$ are indicated on both of its axes.

$$\boxed{\left.\frac{df}{dx}\right|_{x_0} = \frac{1}{\left.\frac{dF}{dx}\right|_{f(x_0)}}} \tag{4.47}$$

- Example

$$y = f(x) = \ln x \qquad \text{i. e.,} \quad x = F(y) = e^y;\ x' = e^y$$
$$y' = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

the well-known result.

4.5 Examples

4.5.1 Sum Rule and Constant Pre-Factor

$$y = x^3 + 7x^2 - 3x - 2$$
$$y' = 3x^2 + 14x - 3$$

$$y = \sin x - \cos x$$
$$y' = \cos x + \sin x$$

4.5.2 Product Rule

$$y = \sin x \cdot \cos x$$
$$y' = \sin x \cdot (-\sin x) + \cos x \cdot \cos x = \cos^2 x - \sin^2 x$$

$$y = e^x \cdot \sin x$$
$$y' = e^x \cdot \cos x + e^x \cdot \sin x = e^x \cdot (\sin x + \cos x)$$

4.5.3 Quotient Rule

$$y = \frac{3x}{5+x}$$
$$y' = \frac{(5+x)\cdot 3 - 3x \cdot 1}{(5+x)^2} = \frac{15}{(5+x)^2}$$

$$y = \tan x = \frac{\sin x}{\cos x}$$
$$y' = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

4.5.4 Reciprocal Function

$$y = \frac{1}{1+x^2} \qquad \text{i. e.,} \quad h(x) = 1 + x^2$$
$$y' = -\frac{2x}{(1+x^2)^2}$$

4.5.5 Chain Rule

$$y = \left(x^3+1\right)^2$$
$$y' = 2 \cdot \left(x^3+1\right) \cdot 3x^2 = 6x^5 + 6x^2$$

$$y = \sqrt{x^2+1}$$
$$y' = \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$$

$$y = \sin(\omega t) \qquad \text{variable, } t$$
$$\dot{y} = \omega \cos(\omega t)$$

$$y = e^{-i\omega t} \qquad \text{variable, } t$$
$$\dot{y} = -i\omega\, e^{-i\omega t}$$

$$y = \sin^2 x = (\sin x)^2$$
$$y' = 2 \sin x \cos x = \sin(2x) \qquad (2^{\text{nd}} \text{ step, addition theorem})$$

$$y = \sin\left(x^2\right)$$
$$y' = \cos\left(x^2\right) \cdot 2x = 2x \cos\left(x^2\right)$$

$$y = e^{ax}$$
$$y' = a\, e^{ax}$$

$$y = e^{f(x)}$$
$$y' = f'(x) \cdot e^{f(x)}$$

4.5.6 Inverse Funktion

$$y = \sqrt{x} \qquad \text{i. e.,} \quad x = y^2;\ x' = 2y$$
$$y' = \frac{1}{2y} = \frac{1}{2\sqrt{x}}$$

$$y = \arcsin x \qquad \text{i. e.,} \quad x = \sin y;\ x' = \cos y$$
$$y' = \frac{1}{\cos y} = \frac{1}{\cos(\arcsin x)}$$

$$= \frac{1}{\sqrt{1 - [\sin(\arcsin x)]^2}}$$
$$= \frac{1}{\sqrt{1 - x^2}}$$

$$y = \arctan x \qquad \text{i. e.,} \quad x = \tan y; \; x' = \frac{1}{\cos^2 y}$$
$$y' = \cos^2 y = [\cos(\arctan x)]^2$$
$$= \frac{1}{1 + [\tan(\arctan x)]^2} \qquad \text{(addition theorem)}$$
$$= \frac{1}{1 + x^2}$$

4.6 Derivatives of Higher Order

4.6.1 Basics

- If the derivative of a function $y = f(x)$ is itself differentiable, we can calculate also its derivative, which is the **second derivative** of $f(x)$. We can write

$$y'' = f''(x) = f^{(2)}(x) = \frac{d^2}{dx^2} f(x)$$
$$= \frac{d}{dx}\left(\frac{d}{dx} f(x)\right) = \frac{d}{dx} f'(x) \tag{4.48}$$

- Derivatives of higher order are calculated in a similar way. The general notation is

$$y^{(n)} = f^{(n)}(x) = \frac{d^n}{dx^n} f(x) \tag{4.49}$$

- Example: n^{th}-order derivative of the power function $y = x^n$

$$y^{(n)} = \frac{d^n}{dx^n} x^n = \frac{d^{n-1}}{dx^{n-1}} \left(n\, x^{n-1}\right) = n \frac{d^{n-1}}{dx^{n-1}} x^{n-1}$$
$$= n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 = n!$$

4.6.2 Example 1: Coefficients of a Power Series

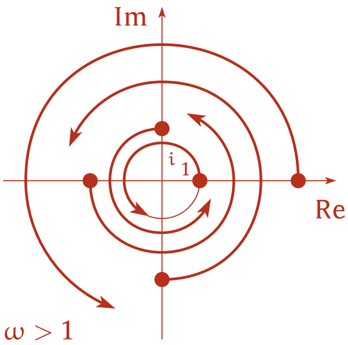
Steady functions can often be expanded as a **power series (Taylor series)** in a certain interval around the expansion point x_0,

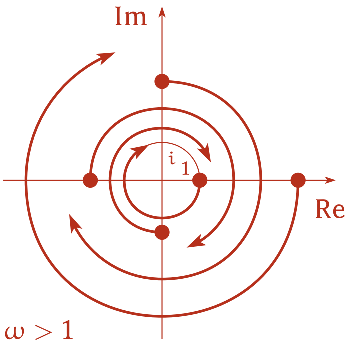
$$f(x_0 + x) = \sum_{n=0}^{\infty} a_n x^n \tag{4.50}$$

(cf. Sections 3.5.2 through 3.5.4). The coefficients a_n are obtained from the n^{th} derivatives of the function at point x_0,

$$a_n = \frac{1}{n!} f^{(n)}(x_0) \tag{4.51}$$

4.6.3 Example 2: The Derivatives of the Complex Exponential Function





- We have discussed the complex exponential function

$$f(t) = e^{i\omega t} \qquad = \cos(\omega t) + i \sin(\omega t)$$

of the real variable t. Its first four derivatives read

$$\begin{aligned} \dot{f}(t) &= i\omega\, e^{i\omega t} &&= \omega\, [-\sin(\omega t) + i\cos(\omega t)] \\ \ddot{f}(t) &= -\omega^2\, e^{i\omega t} &&= \omega^2\, [-\cos(\omega t) - i\sin(\omega t)] \\ f^{(3)}(t) &= -i\omega^3\, e^{i\omega t} &&= \omega^3\, [\sin(\omega t) - i\cos(\omega t)] \\ f^{(4)}(t) &= \omega^4\, e^{i\omega t} &&= \omega^4\, [\cos(\omega t) + i\sin(\omega t)] \quad = \omega^4\, f(t) \end{aligned}$$

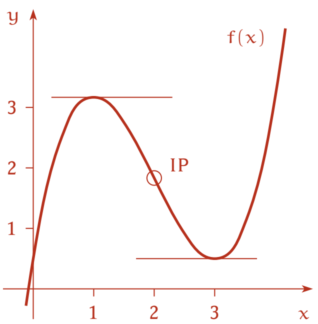
The upper figure shows the function and its derivatives (with increasing radius for arbitrarily chosen $\omega > 1$) in the complex plane for times $t \geqslant 0$. The time zero $t = 0$ is marked with a dot in each function.

- With a minus sign in the exponent, the derivatives read (lower figure)

$$\begin{aligned} g(t) &= e^{-i\omega t} &&= \cos(\omega t) - i\sin(\omega t) \\ \dot{g}(t) &= -i\omega\, e^{-i\omega t} &&= \omega\, [-\sin(\omega t) - i\cos(\omega t)] \\ \ddot{g}(t) &= -\omega^2\, e^{-i\omega t} &&= \omega^2\, [-\cos(\omega t) + i\sin(\omega t)] \\ g^{(3)}(t) &= i\omega^3\, e^{-i\omega t} &&= \omega^3\, [\sin(\omega t) + i\cos(\omega t)] \\ g^{(4)}(t) &= \omega^4\, e^{-i\omega t} &&= \omega^4\, [\cos(\omega t) - i\sin(\omega t)] \\ &= \omega^4\, g(t) \end{aligned}$$

4.7 Applications: Curve Sketching, Problems Involving Extremal Values

4.7.1 Extremal Values and Inflection Points of a Function



- The **extremal values** of a function are its local minima and maxima. The tangent to the function graph is **horizontal** at these points. Hence, the extremal values are equal to the **zeroes of the first derivative** and can be found by calculating the latter.
- For determining whether an extemal value is a local minimum or maximum, we need the second derivative. In the vicinity of a **minimum**, the slope of the tangent increases from negative over zero to positive values, i. e., **the second derivative is positive there**. Similarly, the second derivative is **negative at a local maximum**.
- At the **inflection points (IP)**, the slope of the tangent to the first derivative changes sign. Hence, the inflection points are the **zeroes of the second derivative**, provided that the first derivative is non-zero there.
- If both the first and the second derivative are zero at the same point, different cases can occur. The function can either have an extremal point (example, $y = x^4$ with a local minimum at $x_0 = 0$) or an inflection point with horizontal tangent (example, $y = x^3$ at $x_0 = 0$). For discriminating between these

cases, derivatives of higher order must be considered. For a detailed discussion, the reader is referred to standard textbooks of mathematics.

4.7.2 Example 1: Polynomial of Third Order

- The function plotted on the previous page is the polynomial

$$y = \frac{2}{3}x^3 - 4x^2 + 6x + \frac{1}{2}$$

first derivative

$$y' = 2x^2 - 8x + 6$$

second derivative

$$y'' = 4x - 8$$

- Its extremal points (zeroes of the first derivative) are at

$$x_1 = 1 \qquad \text{and} \qquad x_2 = 3$$

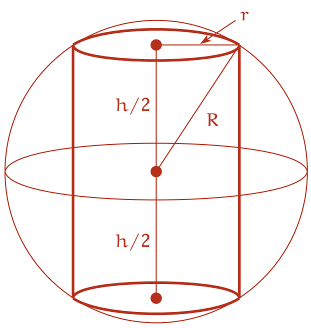
with

$$y''(x_1) = -4 < 0 \qquad \hookrightarrow \qquad \text{maximum}$$
$$y''(x_2) = +4 > 0 \qquad \hookrightarrow \qquad \text{minimum}$$

- There is only one inflection point (zero of the second derivative), located at

$$x_{IP} = 2$$

4.7.3 Example 2: A Problem From Geometry



Calculate the height of the circular cylinder with the biggest volume, which fits into a sphere with given radius R.

- Let h be the height and r the radius of the fitted cylinder. Its volume reads

$$V_{cyl} = r^2 \pi h$$

- The fitting condition connects h and r via Pythagoras's law

$$\left(\frac{h}{2}\right)^2 + r^2 = R^2 \qquad \hookrightarrow \qquad r^2 = R^2 - \frac{h^2}{4}$$

- This lets us express the cylinder volume as a function of only one independent variable, say, h

$$V_{cyl}(h) = \left(R^2 - \frac{h^2}{4}\right)\pi h = R^2 \pi h - \frac{\pi}{4}h^3$$
$$\frac{d}{dh}V_{cyl}(h) = R^2\pi - \frac{3\pi}{4}h^2$$
$$\frac{d^2}{dh^2}V_{cyl}(h) = -\frac{3\pi}{2}h$$

- The first derivative has only one positive zero,

$$\boxed{h_{max} = \frac{2}{\sqrt{3}} R}$$

The second derivative of $V_{cyl}(h)$ is negative for all positive h, so h_{max} is a maximum indeed.

4.8 Functions of More Than One Variable: Partial Derivatives

4.8.1 Basic Ideas

- In this section we discuss functions of more than one independent variable, such as

$$y = f(x,z) \tag{4.52}$$

An example from physics is the pressure of an enclosed quantity of gas, if both the volume V and the temperature T can vary; cf. Section 3.2.1.

- We can calculate the **increment** of the function value corresponding to the increment of one of the variables, while all others are being kept constant. In our example $y = f(x,z)$ this yields

$$\Delta_x y = f(x + \Delta x, z) - f(x,z) \qquad z \text{ constant} \tag{4.53}$$

$$\Delta_z y = f(x, z + \Delta z) - f(x,z) \qquad x \text{ constant} \tag{4.54}$$

- Similarly as for a single variable, we divide these differences by Δx and Δz, respectively, and let the increments tend to zero

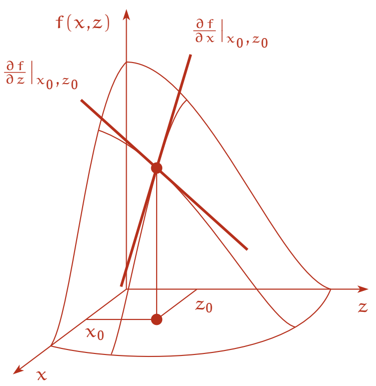
$$\frac{\partial y}{\partial x} = \frac{\partial f(x,z)}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta_x y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, z) - f(x,z)}{\Delta x} \tag{4.55}$$

$$\frac{\partial y}{\partial z} = \frac{\partial f(x,z)}{\partial z} = \lim_{\Delta z \to 0} \frac{\Delta_z y}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(x, z + \Delta z) - f(x,z)}{\Delta z} \tag{4.56}$$

$\frac{\partial y}{\partial x}, \frac{\partial y}{\partial z},$ etc.

- The expressions $\partial y/\partial x$ and $\partial y/\partial z$ are dubbed the **partial derivatives** of the function $y = f(x,z)$. Here the symbol ∂ replaces d for indicating infinitesimal quantities. The calculation rules for partial derivatives are the same as for regular derivatives. **All variables except the one with respect to which we differentiate are treated as constants.**

4.8.2 Graphical Interpretation



- A function of two variables can be graphically depicted as an **expanse in a 3-d coordinate system,** two of its axes representing the two variables and the third one the function value. Similarly, a function of n variables forms a **hypersurface in a space of** $(n+1)$ **dimensions**. A graphical representation is no longer possible for $n > 2$.
- The partial derivatives $\partial f/\partial x|_{x_0,z_0}$ and $\partial f/\partial z|_{x_0,z_0}$ are the slopes of the two lines on the function expanse at position $(x_0; z_0)$, for which either of the variables is constant (z_0 and x_0, respectively) and which intersect in this point. Also for n variables ($n > 2$), each partial derivative represents the slope of a line in the $(n+1)$-d hyperspace.

4.8.3 Example 1: Pressure of an Enclosed Quantity of Gas

In Section 3.2.1 we had written the pressure of an enclosed quantity of one mole of an ideal gas as a function of its volume V and temperature T,

$$p = p(V,T) = R\,\frac{T}{V}$$

where R is the universal gas constant. Its partial derivatives read

$$\frac{\partial p}{\partial T} = \frac{R}{V}$$
$$\frac{\partial p}{\partial V} = -R\,\frac{T}{V^2}$$

4.8.4 Example 2: Plane Waves

- Waves are vibrations propagating in space. The simplest type are plane waves in an isotropic medium. Their wave fronts, i. e., the locations of constant phase, are perpendicular to the propagation direction and have infinite extension.
- A plane wave propagating in x direction can be described by the following complex term, the physically meaningful component being its **real part** (cf. Section 6.3)

  $$A(x,t) = A_0 \exp\left[i(kx - \omega t)\right]$$

  The complex expression is often preferred to sine and cosine functions, since exponentials are easy to calculate with.
- $A(x,t)$ is the **amplitude** of the wave, i. e., its deviation from zero as a function of position and time (e. g., in the case of water waves the height of the rippled water surface and in the case of electromagnetic waves the electrical or magnetic field strength). A_0 is the maximum amplitude, $k = 2\pi/\lambda$ the wave number or the absolute value of the wave vector, and $\omega = 2\pi\nu$ the angular frequency (λ, wavelength; ν, regular frequency). The expression $kx - \omega t$ is dubbed the **phase** of the wave and ω/k is its propagation velocity.

- In the description of wave phenomena, the partial derivatives of the amplitude with respect to space and time are important

$$\frac{\partial A(x,t)}{\partial x} = ik\,A_0 \,\exp\left[i(kx-\omega t)\right] = ik\,A(x,t)$$
$$\frac{\partial A(x,t)}{\partial t} = -i\omega\,A_0 \,\exp\left[i(kx-\omega t)\right] = -i\omega\,A(x,t)$$
$$\frac{\partial^2 A(x,t)}{\partial x^2} = -k^2\,A_0 \,\exp\left[i(kx-\omega t)\right] = -k^2\,A(x,t)$$
$$\frac{\partial^2 A(x,t)}{\partial t^2} = -\omega^2\,A_0 \,\exp\left[i(kx-\omega t)\right] = -\omega^2\,A(x,t)$$

- The prefactor i or $-i$ in the first derivatives indicates that they are **phase-shifted** with respect to $A(x,t)$ by the angles $\pm\pi/2$. This is because $\pm i = \exp(\pm i\pi/2)$. Similarly, the minus sign in the second derivatives indicates the phase shift π (cf. Section 4.6.3).

4.8.5 Example 3: The Gradient

- Let us consider a space-dependent function $f(\vec{r}) = f(x,y,z)$. The **gradient** of f is defined as the vector which has the partial derivatives of the function with respect to the three spatial coordinates as components.[2] It points in the direction of the **strongest variation** of f. We write

grad $f = \vec{\nabla} f$

$$\text{grad}\, f = \vec{\nabla} f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} f \tag{4.57}$$

The symbol $\vec{\nabla}$ is called **nabla operator**; it denotes the vector with the partial spatial derivatives of the following function as components (here f).

- **Special case: Functions with spherical symmetry.** The calculation of the gradient is particularly simple, when the function does not depend on all three spatial coordinates explicitely but only on the distance r from the coordinate origin, i. e., $f = f(r)$. Then we have

$$\frac{\partial f}{\partial x} = \frac{df}{dr}\frac{\partial r}{\partial x} \qquad \text{with } r = \sqrt{x^2+y^2+z^2} \tag{4.58}$$
$$\frac{\partial f}{\partial x} = \frac{df}{dr}\cdot\frac{2x}{2r} = \frac{x}{r}\cdot\frac{df}{dr} \tag{4.59}$$

In the same way

$$\frac{\partial f}{\partial y} = \frac{y}{r}\cdot\frac{df}{dr} \tag{4.60}$$
$$\frac{\partial f}{\partial z} = \frac{z}{r}\cdot\frac{df}{dr} \tag{4.61}$$

[2] Sometimes also the derivative of a function with respect to a single coordinate or even with respect to time is called its spatial or temporal gradient, respectively. This is mathematically not fully correct.

This yields

$$\boxed{\operatorname{grad} f = \frac{df}{dr} \cdot \begin{pmatrix} \frac{x}{r} \\ \frac{y}{r} \\ \frac{z}{r} \end{pmatrix} = \frac{df}{dr} \cdot \frac{\vec{r}}{r} = \frac{df}{dr} \cdot \hat{e}_r} \tag{4.62}$$

$\operatorname{grad} f = \frac{df}{dr} \cdot \hat{e}_r$ for spherical symmetry

In this case the gradient vector is parallel to the corresponding position vector everywhere, i. e., it points radially away from the origin or toward the origin, depending on the sign of df/dr.

- Example: The **electric field strength** is the negative gradient of the electric potential,

  $$\vec{E}(\vec{r}) = -\vec{\nabla}\phi(\vec{r})$$

  The potential around a point charge Q is centrosymmetric with magnitude

  $$\phi(\vec{r}) = \phi(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r}$$

  The value of the corresponding field strength reads

  $$E(r) = -\frac{d\phi}{dr} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}$$

  (cf. Section 4.1.5). Eq. (4.62) indicates that the vector $\vec{E}(\vec{r})$ points toward the charge or away from it everywhere.

- One can calculate the gradient in spherical coordinates (or any other coordinate system) as well. Its components are the partial derivatives with respect to r, θ, and ϕ. We will not discuss this here.

4.9 The Differential

- Consider a function $y = f(x)$ of the independent variable x. At the beginning of this chapter we had discussed finite increments $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ of the variable and the function, respectively. Their quotient $\Delta y/\Delta x$ describes the average slope of the function graph in the interval $(x_1; x_2)$. In the limit $\Delta x \to 0$ it approaches the differential quotient or derivative

  $$y'(x) = \frac{dy}{dx} \tag{4.63}$$

  i. e., the slope of the graph at point x. For $\Delta x \to 0$ we have $x_1 = x_2 = x$.

- It is often interesting to know, how the function value varies at a point x, when the variable is altered only by an infinitesimal amount dx. Such an infinitesimal variation is dubbed the **differential** of x. Since $y'(x)$ is the slope of the function at point x, its differential reads

  $dy = y'(x)\,dx$

  $$\boxed{dy = y'(x)\,dx} \tag{4.64}$$

  Hence, Eq. (4.63) can be formally multiplied by dx.

- For a function which depends on more than one independent variable we can generalize Eq. (4.64) using the partial derivatives. For instance, the differential of $y = y(x,z)$ can be written as

$dy = \frac{\partial y}{\partial x}\,dx + \frac{\partial y}{\partial z}\,dz$ etc.

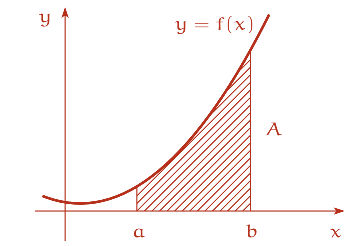
$$\boxed{dy = \frac{\partial y}{\partial x}\,dx + \frac{\partial y}{\partial z}\,dz} \tag{4.65}$$

It is called the **complete differential** of the function $y(x,z)$.

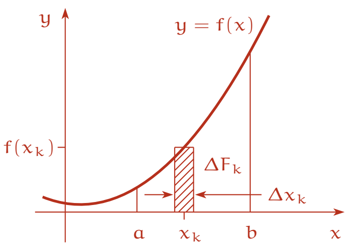
5 Integral Calculus

The basic problem of integral calculus is the **calculation of an area** between the x axis and a function graph, i. e., with one **curvilinear** border in general. We approximate the area by a large number of narrow rectangles, the area of which is easy to calculate. Letting the number of the rectangles tend to infinity and their width to zero then yields the wanted area as the limiting value. Hence, we calculate with **infinitesimal quantities, i. e., differentials** also in this case.

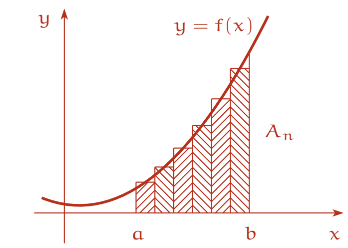
5.1 Introduction: The Definite Integral



- Let us assume, we have a curve which is the graph of a function $f(x)$. Our task is to calculate the area between the curve and the x axis in a given interval $[a; b]$.
- The area can be approximated by dividing the interval into n rectangular stripes with widths $\Delta x_1, \Delta x_2, \ldots, \Delta x_k, \ldots, \Delta x_n$, each centered around a point x_k. The corresponding function values are $f(x_k)$, so the area of the rectangular stripes reads $\Delta F_k = f(x_k)\,\Delta x_k$.



- The sum area of all the stripes yields an approximation A_n for the exact area A. The approximation becomes better with increasing number n of the stripes (and decreasing width Δx_k).



- Finally, in the limit $n \to \infty$, A_n approaches the exact area,

$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta F_k = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x_k \tag{5.1}$$

  The limit is dubbed the **definite integral** of the function $f(x)$ between the borders a and b and is written in the form

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x_k = \int_a^b f(x)\,dx \tag{5.2}$$

  a and b are the **lower** and the **upper integration limit**, respectively.

- From Eqs. (5.1) and (5.2) it follows that

$$A = \int_a^b dF(x) = \int_a^b f(x)\,dx \tag{5.3}$$

  so we have for the differentials

$$\boxed{dF(x) = f(x)\,dx} \quad \text{or} \quad \boxed{f(x) = \frac{dF(x)}{dx}} \tag{5.4}$$

  [cf. Eqs. (4.63), (4.64)].

- The calculation of the unknown area A leads to the problem of **finding a new function $F(x)$ for a given function $f(x)$, so $f(x)$ is the derivative of $F(x)$**. Hence, differentiation and integration are inverse mathematical operations.

- We will see later in Section 5.6, how the area A is obtained using the function $F(x)$. Prior to that we want to find rules for calculating $F(x)$.

5.2 Indefinite Integral, Primitive, Antiderivative

- Our goal consists in finding a function $F(x)$ which, upon differentiation, yields the integrand function $f(x)$,

$$F'(x) = f(x) \tag{5.5}$$

- $F(x)$ is dubbed **indefinite integral**, **primitive**, or **antiderivative** of $f(x)$.

- From differential calculus we know that two functions, which differ only by an additive constant, have identical derivatives. Consequently, **a constant number can always be added to an antiderivative**. We write

$$\int f(x)\,dx = F(x) + C \tag{5.6}$$

Integration constant C

The constant C is dubbed **integration constant** and can be **freely chosen**. The function $f(x)$ is the **integrand**. Antiderivatives are indicated by **omitting the integration limits** at the integral sign.

- To an elementary function $f(x)$, the antiderivative can often be obtained by "guessing" as we know the differentiation rules. This is legitimate, if we cross-check the result by calculating the derivative. Examples

$$f(x) = x \quad \hookrightarrow \quad \int x\,dx = \frac{x^2}{2} + C$$

$$\hookrightarrow \quad \frac{d}{dx}\left(\frac{x^2}{2} + C\right) = x$$

$$f(x) = \cos x \quad \hookrightarrow \quad \int \cos x\,dx = \sin x + C$$

$$\hookrightarrow \quad \frac{d}{dx}(\sin x + C) = \cos x$$

$$f(x) = e^x \quad \hookrightarrow \quad \int e^x\,dx = e^x + C$$

$$\hookrightarrow \quad \frac{d}{dx}(e^x + C) = e^x$$

5.3 Integrals of Elementary Functions

5.3.1 Small Look-Up Table

$$f(x) = a = \text{const.} \qquad \int a\,dx = ax + C \tag{5.7}$$

$$f(x) = x^n \qquad \int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \qquad \boxed{n \neq -1} \tag{5.8}$$

$$f(x) = \frac{1}{x} \qquad \int \frac{1}{x}\,dx = \ln|x| + C \qquad \boxed{x \neq 0} \tag{5.9}$$

$$f(x) = \sin x \qquad \int \sin x\,dx = -\cos x + C \tag{5.10}$$

$$f(x) = \cos x \qquad \int \cos x\,dx = \sin x + C \tag{5.11}$$

$$f(x) = \frac{1}{\cos^2 x} \qquad \int \frac{1}{\cos^2 x}\,dx = \tan x + C \tag{5.12}$$

$$f(x) = \frac{1}{\sin^2 x} \qquad \int \frac{1}{\sin^2 x}\,dx = -\cot x + C \tag{5.13}$$

$$f(x) = e^x \qquad \int e^x\,dx = e^x + C \tag{5.14}$$

$$f(x) = a^x \qquad \int a^x\,dx = \frac{1}{\ln a}\,a^x + C \tag{5.15}$$

$$f(x) = \frac{1}{\sqrt{1-x^2}} \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C \tag{5.16}$$

$$f(x) = \frac{1}{1+x^2} \qquad \int \frac{1}{1+x^2}\,dx = \arctan x + C \tag{5.17}$$

5.3.2 Examples

$$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{1}{1+\frac{1}{2}}\,x^{1+1/2} + C = \frac{2}{3}\sqrt{x^3} + C$$

$$\int \frac{1}{\sqrt{x}}\,dx = \int x^{-1/2}\,dx = \frac{1}{1-\frac{1}{2}}\,x^{1-1/2} + C = 2\sqrt{x} + C$$

$$\int x^{5/3}\,dx = \frac{1}{1+\frac{5}{3}}\,x^{1+5/3} + C = \frac{3}{8}\sqrt[3]{x^8} + C$$

5.4 Rules for Composite Functions

5.4.1 Sum Rule

$$\boxed{\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx} \tag{5.18}$$

As proof differentiate both sides.

5.4.2 Constant Pre-Factor

$$\boxed{\int a \cdot f(x)\,dx = a \cdot \int f(x)\,dx} \tag{5.19}$$

As proof differentiate both sides.

5.4.3 Integration by Parts

There is no general rule for calculating the integral of products of two or more functions. If, however, the integrand can be written as the product of one function with the derivative of another, we have

$$\boxed{\int f(x) \cdot g'(x)\,dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x)\,dx} \tag{5.20}$$

If we are lucky, the integral on the right-hand side is easier to perform than the original one. The integration by parts follows from the product rule for derivatives,

$$\frac{d}{dx}\left[f(x) \cdot g(x)\right] = f'(x) \cdot g(x) + f(x) \cdot g'(x) \qquad \hookrightarrow$$

$$\underbrace{\int \frac{d}{dx}\left[f(x) \cdot g(x)\right] dx}_{f(x) \cdot g(x)} = \int f'(x) \cdot g(x)\,dx + \int f(x) \cdot g'(x)\,dx$$

5.4.4 Integration by Substitution

Assume, an integral has the form

$$\int f\left[g(x)\right] \cdot g'(x)\,dx$$

In this case we can define a substitution variable $u = g(x)$ with the differential $du = g'(x)\,dx$ and obtain

$$\boxed{\int f\left[g(x)\right] \cdot g'(x)\,dx = \int f(u)\,du} \tag{5.21}$$

This equation follows from the chain rule of differentiation: Let $F(u)$ be a primitive of $f(u)$, i. e., $dF/du = f(u)$. Its derivative with respect to x reads

$$\frac{dF}{dx} = \frac{dF}{du}\frac{du}{dx} = \frac{dF}{du}\frac{dg}{dx} = f(u)\,g'(x) = f\left[g(x)\right] \cdot g'(x)$$

Integrating the first and the last expression with respect to x yields the substitution rule [Gl. (5.21)],

$$\int \frac{dF}{dx}\,dx = F(u) + C = \int f(u)\,du = \int f\left[g(x)\right] \cdot g'(x)\,dx$$

5.5 Examples

5.5.1 Sum Rule and Constant Pre-Factor

$$\int 3\cos x \, dx = 3\sin x + C$$

$$\int \pi\, e^x \, dx = \pi\, e^x + C$$

$$\int \left(x^{27} + \sin x\right) dx = \frac{1}{28}\, x^{28} - \cos x + C$$

$$\int \left[A\left(3x^2 + B\right)^2 + \cos x\right] dx = \frac{9}{5}\, A x^5 + 2ABx^3 + AB^2 x + \sin x + C$$

5.5.2 Integration by Parts

$$\int \ln x \, dx = \int \underbrace{1}_{g'(x)} \cdot \underbrace{\ln x}_{f(x)} \, dx = ?$$

$$g(x) = x; \quad f'(x) = \frac{1}{x} \quad \hookrightarrow$$

$$\int \ln x \, dx = x \ln x - \int \frac{x}{x} \, dx = \boxed{x\,(\ln x - 1) + C}$$

$$\int \underbrace{x}_{f(x)} \underbrace{\sin x}_{g'(x)} \, dx = ?$$

$$g(x) = -\cos x; \quad f'(x) = 1 \quad \hookrightarrow$$

$$\int x \sin x \, dx = -x \cos x + \int \cos x \, dx = \boxed{-x \cos x + \sin x + C}$$

$$\int \underbrace{x^2}_{f(x)} \underbrace{e^x}_{g'(x)} \, dx = ?$$

$$g(x) = e^x; \quad f'(x) = 2\,x \quad \hookrightarrow$$

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int \underbrace{x}_{f(x)} \underbrace{e^x}_{g'(x)} \, dx$$

$$g(x) = e^x; \quad f'(x) = 1 \quad \hookrightarrow$$

$$\int x^2 e^x \, dx = x^2 e^x - 2\left(x\, e^x - \int e^x \, dx\right) = \boxed{e^x \left(x^2 - 2x + 2\right) + C}$$

The integration by parts has been applied twice here.

$$\int \cos^2 x \, dx = \int \underbrace{\cos x}_{f(x)} \underbrace{\cos x}_{g'(x)} \, dx = ?$$

$$g(x) = \sin x; \quad f'(x) = -\sin x \quad \hookrightarrow$$

$$\int \cos^2 x \, dx = \sin x \, \cos x - \int (-\sin x) \sin x \, dx =$$

$$\sin x \, \cos x + \int \left(1 - \cos^2 x\right) dx$$

The two integrals of the function $\cos^2 x$ on the left- and the right-hand side of the equation can be combined,

$$2 \int \cos^2 x \, dx = \sin x \, \cos x + x + C^* \quad \hookrightarrow$$

$$\int \cos^2 x \, dx = \frac{1}{2} (\sin x \, \cos x + x + C^*) = \boxed{\frac{1}{4} \sin (2x) + \frac{1}{2} x + C}$$

In the last step, one of the addition theorems has been used again ($C^* = 2C$).

5.5.3 Integration by Substitution

$$\int 3 \, e^{3x} \, dx = ?$$

$$\text{subst.: } 3x = u; \quad 3 \, dx = du \quad \hookrightarrow$$

$$\int 3 \, e^{3x} \, dx = \int e^u \, du = e^u + C = \boxed{e^{3x} + C}$$

$$\int e^{-i\omega t} \, dt = ?$$

$$\text{subst.: } -i\omega t = u; \quad -i\omega \, dt = du \quad \hookrightarrow$$

$$\int e^{-i\omega t} \, dt = \frac{i}{\omega} \int e^u \, du = \frac{i}{\omega} e^u + C = \boxed{\frac{i}{\omega} e^{-i\omega t} + C}$$

$$\int x^2 \cos\left(x^3\right) \, dx = ?$$

$$\text{subst.: } x^3 = u; \quad 3x^2 \, dx = du \quad \hookrightarrow$$

$$\int x^2 \cos\left(x^3\right) \, dx = \frac{1}{3} \int \cos u \, du = \frac{1}{3} \sin u + C = \boxed{\frac{1}{3} \sin\left(x^3\right) + C}$$

$$\int (3x + 7)^{27} \, dx = ?$$

$$\text{subst.: } 3x + 7 = u; \quad 3 \, dx = du \quad \hookrightarrow$$

$$\int (3x+7)^{27}\,dx = \frac{1}{3}\int u^{27} du = \frac{1}{84}\,u^{28} + C = \boxed{\frac{1}{84}\,(3x+7)^{28} + C}$$

$$\int \left(x^3+2\right)^2 \cdot 3x^2\,dx = ?$$

$$\text{subst.: } x^3+2 = u; \quad 3x^2\,dx = du \quad \hookrightarrow$$

$$\int \left(x^3+2\right)^2 \cdot 3x^2\,dx = \int u^2 du = \frac{1}{3}\,u^3 + C = \boxed{\frac{1}{3}\left(x^3+2\right)^3 + C}$$

The integral can also be performed by expanding the brackets or using integration by parts. Substitution is the fastest way.

$$\int \frac{dx}{x\,\ln x} = ?$$

$$\text{subst.: } \ln x = u; \quad \frac{1}{x}\,dx = du \quad \hookrightarrow$$

$$\int \frac{dx}{x\,\ln x} = \int \frac{du}{u} = \ln|u| + C = \boxed{\ln|\ln x| + C}$$

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = ?$$

$$\text{subst.: } \sin x = u; \quad \cos x\,dx = du \quad \hookrightarrow$$

$$\int \cot x\,dx = \int \frac{du}{u} = \ln|u| + C = \boxed{\ln|\sin x| + C}$$

$$\int \sin^3 x\,dx = ?$$

To get the differential right, we must substitute $\cos x$ rather than $\sin x$ in this case,

$$\text{subst.: } \cos x = u; \quad -\sin x\,dx = du \quad \hookrightarrow$$

$$\int \sin^3 x\,dx = \int \left(1-\cos^2 x\right)\sin x\,dx = -\int \left(1-u^2\right)du =$$

$$\int \left(u^2-1\right)du = \frac{1}{3}\,u^3 - u + C = \boxed{\frac{1}{3}\,\cos^3 x - \cos x + C}$$

5.5.4 Concluding Remarks

- The above examples—in particular the last one—have shown that the best way for calculating indefinite integrals is not always visible at first glance. Finding the fastest way is a matter of practice. There is no general recipe.

Not every integral is doable!

- Moreover, not every integral can be calculated in closed form. Counterexample

$$F(x) = \int \frac{e^{-x}}{x}\,dx$$

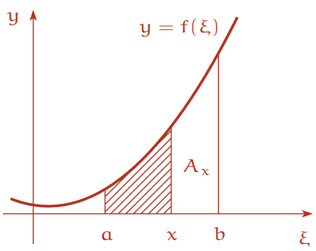
All we can do is expand the integrand into its power series and integrate it term by term.

- Many integrals can be found in mathematical formularies, e. g., Bronstein, Semendjajew, *Taschenbuch der Mathematik*, publisher Harri Deutsch.

5.6 Definite and Indefinite Integral

5.6.1 Area Calculation with the Antiderivative

- We had introduced the definite integral for the purpose of calculating the area underneath a function graph [$y = f(x)$] between the bounds a and b (cf. Section 5.1).



- Let us start by considering only a part A_x of this area, between a and a value x (with $a < x < b$),

$$A_x = \int_a^x f(\xi)\,d\xi = \int_a^x dF(\xi) = F(x) + C \tag{5.22}$$

Since x is one of the integration bounds, the variable has been changed to ξ.

- The area changes upon shifting x; hence the antiderivative F on the right-hand side must have x as its variable.

- When we shift x to the lower integration limit a, the area becomes zero,

$$\lim_{x\to a} A_x = \lim_{x\to a} \int_a^x f(\xi)\,d\xi = F(a) + C = 0 \tag{5.23}$$

Hence, the integration constant is related to the value of the antiderivative at position a,

$$\boxed{C = -F(a)} \tag{5.24}$$

- The area between the limits a and x then reads [see Eqs. (5.22) and (5.24)]

$$\int_a^x f(\xi)\,d\xi = F(x) - F(a) \tag{5.25}$$

and, accordingly, the whole area between a und b

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{5.26}$$

- Consequence: **If we know any antiderivative $F(x)$ of $f(x)$ with arbitrary integration constant C, the definite integral is obtained as**

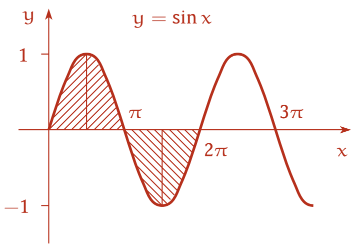
$$\boxed{\int_a^b f(x)\,dx = F(x)\Big|_a^b = \Big[F(x)\Big]_a^b = F(b) - F(a)} \tag{5.27}$$

- Example

$$\int_a^b x^2\,dx = \left[\frac{x^3}{3} + C\right]_a^b = \frac{b^3}{3} - \frac{a^3}{3} = \boxed{\frac{1}{3}\left(b^3 - a^3\right)}$$

- Note:
  - Areas below the x axis have a negative sign.
  - The sign of an area changes when the upper integration bound is located left of the lower one.

5.6.2 Example: Sine Function



$$\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = -\cos\pi - (-\cos 0) = 1 + 1 = 2$$

The area underneath a sine arc has the integer value of 2 area units. Analogously

$$\int_\pi^{2\pi} \sin x\,dx = -\cos x\Big|_\pi^{2\pi} = -\cos(2\pi) + \cos\pi = -1 - 1 = -2$$

$$\int_0^{2\pi} \sin x\,dx = -\cos x\Big|_0^{2\pi} = -\cos(2\pi) + \cos 0 = -1 + 1 = 0$$

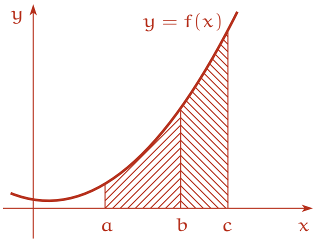
$$\int_{\pi/2}^{3\pi/2} \sin x\,dx = -\cos x\Big|_{\pi/2}^{3\pi/2} = -\cos\frac{3\pi}{2} + \cos\frac{\pi}{2} = 0 + 0 = 0$$

In the calculation of definite integrals, the integration constant can be omitted (or set equal to zero), since it cancels out upon inserting the bounds anyway.

5.7 Rules for Definite Integrals

Most of these rules are immediately clear or follow easily from those for indefinite integrals. The only exception is the substitution rule, since **substitution of the variable affects the integration bounds**.

5.7.1 Adding Areas



$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx \tag{5.28}$$

and, conversely,

$$\int_a^c f(x)\,dx - \int_b^c f(x)\,dx = \int_a^b f(x)\,dx \tag{5.29}$$

$$\int_a^a f(x)\,dx = 0 \tag{5.30}$$

(immediately clear).

5.7.2 Swapping the Integration Bounds

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \tag{5.31}$$

5.7.3 Differentiating with Respect to One of the Bounds

$$\frac{d}{dx}\int_a^x f(\xi)\,d\xi = \frac{d}{dx}\left[F(x) - F(a)\right] = f(x) \tag{5.32}$$

5.7.4 Integration by Parts

$$\int_a^b f(x)\cdot g'(x)\,dx = f(x)\cdot g(x)\Big|_a^b - \int_a^b f'(x)\cdot g(x)\,dx \tag{5.33}$$

with

$$f(x)\cdot g(x)\Big|_a^b = f(b)\cdot g(b) - f(a)\cdot g(a)$$

5.7.5 Integration by Substitution

$$\int_a^b f\left[g(x)\right]\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du \tag{5.34}$$

example

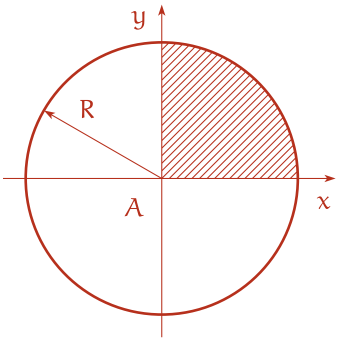
$$\int_1^2 \exp(3x)\ dx = \frac{1}{3}\int_3^6 \exp(u)\ du = \frac{1}{3}\exp(u)\Big|_3^6 = \frac{1}{3}\left(e^6 - e^3\right)$$

or

$$\int_1^2 \exp(3x)\ dx = \frac{1}{3}\exp(3x)\Big|_1^2 = \frac{1}{3}\left(e^6 - e^3\right)$$

Attention: If we keep the substituted variable after calculating the antiderivative, the integration bounds must be adapted. If we re-substitute the original variable, the original bounds are correct.

5.7.6 Example: Area of a Circle



- We place the coordinate origin at circle center. It suffices to calculate the area of the first quadrant and multiply the result by four.
- The equation for the circle line reads

$$x^2 + y^2 = R^2 \qquad \hookrightarrow \qquad y = \sqrt{R^2 - x^2}$$

for $0 \leqslant x \leqslant R$.

- Hence, the circle area is calculated as

$$A_{\text{circle}} = 4 \int_0^R \sqrt{R^2 - x^2}\, dx = 4\, R \int_0^R \sqrt{1 - \left(\frac{x}{R}\right)^2}\, dx$$

$$\text{subst.: } \frac{x}{R} = \sin u; \quad dx = R \cos u\, du \quad \hookrightarrow$$

$$A_{\text{circle}} = 4\, R^2 \int_0^{\pi/2} \sqrt{1 - \sin^2 u}\, \cos u\, du = 4\, R^2 \int_0^{\pi/2} \cos^2 u\, du$$

$$= 4\, R^2 \left[\frac{1}{2} u + \frac{1}{4} \sin(2\, u)\right]_0^{\pi/2}$$

(cf. the last example in Section 5.5.2)

$$A_{\text{circle}} = 4\, R^2 \cdot \frac{\pi}{4} = \boxed{R^2 \pi}$$

- Calculating the circle area in this way is tedious. In particular, the required substitution is not obvious. The calculation becomes much easier and shorter, if we perform it—corresponding to the geometry of the problem—in polar rather than Cartesian coordinates. This will be demonstrated in Section 5.9.2.

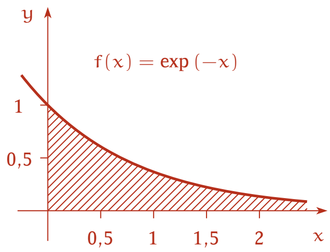
5.8 Infinite Integration Limits

- We want to calculate an integral with **infinite upper integration limit**,

$$\int_a^\infty f(x)\, dx = ?$$

- Solution: Calculate the integral with a **finite** upper bound b and, subsequently, determine the limit for $b \to \infty$,

$$\int_a^\infty f(x)\, dx = \lim_{b \to \infty} \int_a^b f(x)\, dx \qquad (5.35)$$



- Example: Exponential function

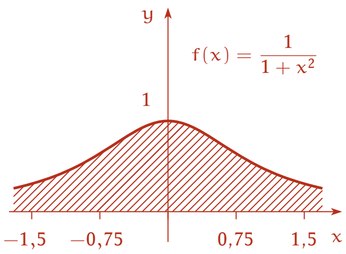
$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}\right]_0^b$$

$$= 1 - \lim_{b\to\infty} e^{-b} = \boxed{1}$$

The infinitely long, ever narrowing, area between the coordinate axes and the graph of the exponential has the smooth value of 1 unit area.

- If the lower integration limit is at minus infinity, we proceed in a similar way,

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx \tag{5.36}$$



- Simple example with two infinite bounds

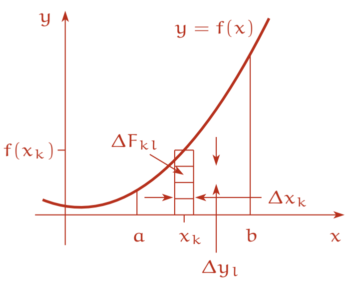
$$\int_{-\infty}^{+\infty}\frac{1}{1+x^2}\,dx = \arctan x\Big|_{-\infty}^{+\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \boxed{\pi}$$

The bell-shaped curve representing the integrand is dubbed "Lorentzian (curve)". It plays an important role in the emission and absorption of electromagnetic waves, e. g., light. The area underneath it has the size π unit area.

5.9 Surface Integrals in Polar Coordinates

5.9.1 Motivation: The Surface Differential in Cartesian Coordinates

- The goal of integral calculus is the calculation of the area underneath a function graph $f(x)$. So far, we had divided it into narrow rectangles of width Δx_k and height $f(x_k)$. The area beneath the curve is the sum of all the rectangular areas $\Delta F_k = f(x_k)\,\Delta x_k$ for $\Delta x_k \to 0$.



- The sought-after area can be subdivided further by assembling each narrow element of short, vertically stacked, rectangles of height Δy_l (and width Δx_k). If we let Δx_k and Δy_l tend to zero, we obtain **two** integrations,

$$A = \int_a^b dx \int_0^{f(x)} dy = \int_a^b dx\; y\Big|_0^{f(x)} = \int_a^b f(x)\,dx \tag{5.37}$$

The (trivial) y integral must be performed first, since its result yields the integrand of the x integration.

- A small (but not infinitesimal) rectangular surface element in Cartesian coordinates has the area

$$\Delta F_{kl} = \Delta x_k\,\Delta y_l \tag{5.38}$$

so the **surface differential** reads

$dF = dx\,dy$

$$dF = dx\,dy \tag{5.39}$$

- This procedure allows us to calculate areas of arbitrary shape in plane Cartesian coordinates as a **double integral** along the coordinate axes,

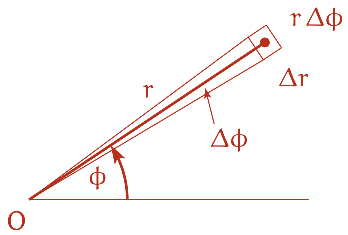
$$\boxed{A = \iint\limits_{(A)} dx\, dy} \tag{5.40}$$

The integration bounds must be chosen such that the area A is covered [indicated by (A) below the integral sign].

- Simple example: The area of a rectangle of width W in x direction and length L in y direction is

$$A_{\text{rectangle}} = \int_0^W dx \int_0^L dy = x\Big|_0^W \; y\Big|_0^L = \boxed{W L}$$

5.9.2 Changing to Polar Coordinates



$dF = r\, dr\, d\phi$

- It is always possible to compose an arbitrary area of small rectangular-shaped elements. This works in a polar coordinate system as well. Let us consider a position $\vec{r} = (r; \phi)$.
- A small increment in radial direction is simply Δr. In tangential direction, on the other hand, a length element is given by the correponding angular increment $\Delta\phi$ and the distance from the origin and reads $r\,\Delta\phi$.
- This yields the **surface differential in polar coordinates** as

$$dF = r\, dr\, d\phi \tag{5.41}$$

The slight curvature of the segment $r\, d\phi$ becomes unimportant when we go over to infinitesimal quantities. An arbitrary area is then calculated as

$$\boxed{A = \iint\limits_{(A)} r\, dr\, d\phi} \tag{5.42}$$

- Example: The area of a circle with radius R is calculated very simply in polar coordinates,

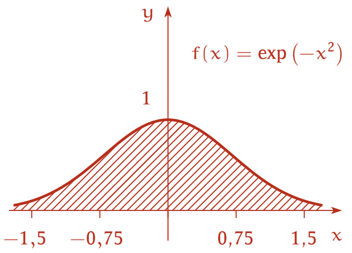
$$A_{\text{circle}} = \int_0^R r\, dr \int_0^{2\pi} d\phi = \frac{1}{2} R^2 \cdot 2\pi = \boxed{R^2 \pi}$$

Again, it is important to choose the integral limits so as to cover the required area.

5.9.3 Application: The Area Beneath a Gaussian

- The Gaussian normal distribution plays an important role for describing the statistical fluctuations of data, e. g., results of a measurement, around their average. In its simplest form it reads

$$f(x) = \exp\left(-x^2\right) \tag{5.43}$$



- Its graph is a bell-shaped curve, similar to a Lorentzian, but decaying much faster in its wings (cf. p. 66). Its primitive, the error function, cannot be written down in closed form, only as a power series. Nevertheless, it is possible to analytically calculate the definite integral from $-\infty$ to $+\infty$. To this end we write the integral twice, choosing x and y as variable:

$$A_{\text{Gauss}} = \int_{-\infty}^{+\infty} \exp\left(-x^2\right) dx = \int_{-\infty}^{+\infty} \exp\left(-y^2\right) dy \tag{5.44}$$

- Multiplying both expressions yields

$$A^2_{\text{Gauss}} = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \, \exp - \left[\left(x^2 + y^2\right)\right] \tag{5.45}$$

- The variables x and y can be viewed as Cartesian coordinates in a plane, where the integration is to be performed over the whole plane. Changing to polar coordinates leads to

$$A^2_{\text{Gauss}} = \int_0^{2\pi} d\phi \int_0^{\infty} dr \, r \, \exp\left(-r^2\right) \tag{5.46}$$

- With the substitution $r^2 = u$, the r integral can be easily performed (cf. p. 66) yielding

$$A^2_{\text{Gauss}} = 2\pi \, \frac{1}{2}; \qquad \text{hence} \quad A_{\text{Gauss}} = \boxed{\sqrt{\pi}}$$

Due to the faster decay, the area is smaller than the area underneath a Lorentzian of the same height.

5.10 Volume Integrals

In a similar way as areas, also volumes of given bodies can be calculated by integration, if we compose them of **small cuboid elements**. Since we are now dealing with three-dimensional space, **three** integrations along the coordinates of space are required.

5.10.1 Cartesian Coordinates

- A two-dimensional Cartesian coordinate system is extended to three dimensions by adding the z axis in the direction perpendicular to the xy plane.
- Hence, in Cartesian coordinates the **volume differential** simply reads

$dV = dx\,dy\,dz$

$$dV = dx\,dy\,dz \tag{5.47}$$

and a volume is calculated as

$$\boxed{V = \iiint_{(V)} dx\,dy\,dz} \tag{5.48}$$

- Example: A cuboid of width W (along the x axis), length L (along the y axis), and height H (along the z axis) has the volume

$$V_{\text{cuboid}} = \int_0^B dx \int_0^L dy \int_0^H dz = x\Big|_0^B \, y\Big|_0^L \, z\Big|_0^H = \boxed{B\,L\,H}$$

5.10.2 Cylindrical Coordinates

- Cylindrical coordinates are obtained by adding the z axis perpendicularly to the $\rho\,\phi$ plane of a 2-d polar coordinate system (cf. Section 1.3.2).
- Hence, the volume differential is again obtained very easily from the surface differential in polar coordinates as

$dV = \rho\, d\rho\, d\phi\, dz$

$$dV = \rho\, d\rho\, d\phi\, dz \tag{5.49}$$

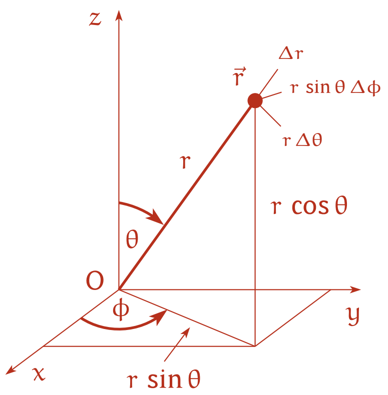
and the expression for a volume reads

$$\boxed{V = \iiint_{(V)} \rho\, d\rho\, d\phi\, dz} \tag{5.50}$$

- Example: Volume of an upright circular cylinder with radius R and height H

$$V_{\text{cylinder}} = \int_0^R \rho\, d\rho \int_0^{2\pi} d\phi \int_0^H dz = \frac{1}{2} R^2 \cdot 2\pi \cdot H = \boxed{R^2 \pi H}$$

5.10.3 Spherical Coordinates



$dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$

- Let us consider a small, but finite, cuboid volume element ΔV in spherical coordinates at position $\vec{r} = (r;\, \theta;\, \phi)$. It is spanned by the following three increments.
- The increment in radial direction is Δr.
- The increment in tangential direction given by an angular increment $\Delta\theta$ and constant ϕ is $r\,\Delta\theta$.
- The increment in the other tangential direction given by an angular increment $\Delta\phi$ and constant θ is $r \sin\theta\, \Delta\phi$. It corresponds to the increment $\rho\,\Delta\phi$ in cylindrical coordinates.
- Multiplication of these three increment lengths and moving to infinitesimal quantities yields the volume differential

$$dV = r^2 \sin\theta\, dr\, d\theta\, d\phi \tag{5.51}$$

and the expression for a volume

$$\boxed{V = \iiint_{(V)} r^2 \sin\theta\, dr\, d\theta\, d\phi} \tag{5.52}$$

- Example: A sphere with radius R has the volume

$$V_{\text{sphere}} = \int_0^R r^2 dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi = \frac{1}{3} R^3 \cdot 2 \cdot 2\pi = \boxed{\frac{4\pi}{3} R^3}$$

Please note: In order to cover the complete volume of the sphere, the **polar angle θ must vary from 0 to π** (i. e., from the north pole via the equator to the south pole) and the **azimuth angle ϕ over the full longitude range from 0 to 2π**.

5.10.4 Integrals over the Surface of a Sphere, Solid Angle

- The surface of a sphere can be calculated by setting $r = R =$ const. and performing only the integrals over the angle coordinates. To this end we must first define the surface element in spherical coordinates,

$dA = R^2 \sin\theta \, d\theta \, d\phi$

$$dA = R^2 \sin\theta \, d\theta \, d\phi \tag{5.53}$$

The length differential dr in radial direction is not present, since $r =$ const. Hence, the surface of a sphere is calculated as

$$\boxed{A_{sph-sf} = R^2 \iint\limits_{(A)} \sin\theta \, d\theta \, d\phi} \tag{5.54}$$

and has the value

$$A_{sph-sf} = R^2 \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi = R^2 \cdot 2 \cdot 2\pi = \boxed{4\pi R^2}$$

solid-angle element $d\Omega$

$d\Omega = \sin\theta \, d\theta \, d\phi$

- The surface element of the **unit sphere** (i. e., a sphere with radius $R = 1$) is also called **solid-angle elelement $d\Omega$**; it reads

$$d\Omega = \sin\theta \, d\theta \, d\phi \tag{5.55}$$

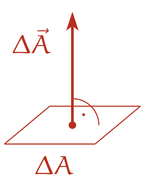
and the integration over a solid angle Ω is

$$\boxed{\Omega = \iint\limits_{(\Omega)} \sin\theta \, d\theta \, d\phi} \tag{5.56}$$

For the full solid angle (i. e., the total surface of the unit sphere) we obtain the result 4π.

- Example: The **electric flux** of a point charge through a spherical surface. We place an electrical point charge Q at the center of a sphere with radius R and calculate the electric flux $\Phi_{el.}$ of its field through the surface of the sphere. The flux is defined as

$$\Phi_{el} = \iint\limits_{(A)} \vec{E} \cdot d\vec{A}$$



To any surface element ΔA (or dA), one can ascribe a vector $\Delta\vec{A}$ (or $d\vec{A}$). Its absolute value is equal to the size of the surface element and its direction is perpendicular to it. Hence, the vectors of all the surface elements of a sphere point radially outward—as does the field-strength vector. The scalar product is then equal to the product of the absolute values,

$$\Phi_{el} = \iint\limits_{(A)} E \cdot dA = R^2 \int\limits_0^{\pi} \sin\theta \, d\theta \int\limits_0^{2\pi} d\phi \, \frac{Q}{4\pi\,\epsilon_0\,R^2}$$

$$= \frac{Q}{4\pi\,\epsilon_0} \int\limits_0^{\pi} \sin\theta \, d\theta \int\limits_0^{2\pi} d\phi = \boxed{\frac{Q}{\epsilon_0}}$$

The electric flux is always Q/ϵ_0, independent of the size of the sphere.

5.10.5 Example: Moments of Inertia

The last example has already demonstrated that two- and three-dimensional integrals cannot only be used for calculating surfaces and volumes, but that other quantities requiring integrations over surfaces or volumes are accessible as well. Here we consider the **moments of inertia** of a solid cylinder and a solid sphere for rotation about the respective symmetry axis.

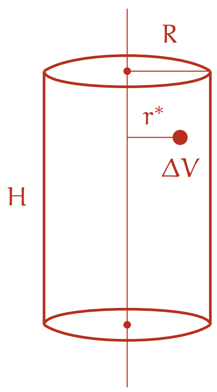
- The moment of inertia Θ plays a similar role for the **rotation** of a body as its mass does for translational motion. In contrast to the mass, the moment of inertia depends on the shape of the body, the distribution of its masses, and even on the choice of the rotation axis. It is defined as

$$\boxed{\Theta = \iiint\limits_{(M)} (r^*)^2 \, dm} \qquad (5.57)$$

  where r^* is the **perpendicular distance** of mass element dm from the rotation axis. The integral is to be performed over the complete mass M of the body.

- In a homogeneous body, the mass element is the product of the mass density η and the respective volume element dV, so $dm = \eta\, dV$. Hence, the mass integral reduces to a volume integral,

$$\Theta = \eta \iiint\limits_{(V)} (r^*)^2 \, dV \qquad (5.58)$$



- Solid cylinder with radius R, height H, and mass M: We use cylindrical coordinates with the cylinder axis, which is also the rotation axis, coinciding with the z axis of the coordinate system. Then the distance r^* is equal to the coordinate ρ for all volume elements, and the moment of inertia is

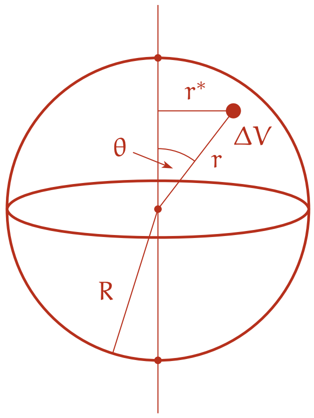
$$\Theta_{cylinder} = \eta \int\limits_0^{R} \rho^3 \, d\rho \int\limits_0^{2\pi} d\phi \int\limits_0^{H} dz = \eta \cdot \frac{1}{4} R^4 \cdot 2\pi \cdot H$$

$$= \frac{1}{2} R^2 \cdot \eta\, R^2\, \pi\, H$$

  With the cylinder mass $M = \eta\, V = \eta\, R^2\, \pi\, H$ we can write

$$\boxed{\Theta_{cylinder} = \frac{1}{2} M\, R^2}$$

  For comparison: A thin-walled hollow cylinder, in which all the mass elements are located at the same distance R from the rotation axis, has twice this moment of inertia, viz. $\Theta_{h-cyl} = M R^2$.



- Solid sphere with radius R and mass M: Here we use spherical coordinates with the origin at the center of the sphere. The rotation axis is the z axis again. The perpendicular distance from it is $r^* = r \sin\theta$, so

$$\Theta_{\text{sphere}} = \eta \int_0^R r^4 \, dr \int_0^\pi \sin^3\theta \, d\theta \int_0^{2\pi} d\phi$$
$$= \eta \cdot \frac{1}{5} R^5 \cdot \left[\frac{1}{3}\cos^3\theta - \cos\theta\right]_0^\pi \cdot 2\pi$$

The antiderivative of $f(x) = \sin^3 x$ has been calculated in Section 5.5.3. Inserting the limits 0 and π finally yields

$$\Theta_{\text{sphere}} = \eta \cdot \frac{1}{5} R^5 \cdot \frac{4}{3} \cdot 2\pi = \frac{2}{5} R^2 \cdot \eta \frac{4\pi}{3} R^3$$

or, with the mass of the sphere $M = \eta\,(4\pi/3)\,R^3$,

$$\boxed{\Theta_{\text{sphere}} = \frac{2}{5} M R^2}$$

For a thin-walled hollow sphere we obtain $\Theta_{h-sph} = (2/3)\, M R^2$. The verification is left to the reader as a problem.

6 Simple Examples of Ordinary Differential Equations

6.1 Growth of a Population

We assume that some population (e. g., a culture of bacteria) comprises $N(t)$ individuals at time t. We wish to figure out, how this number changes as a function of time.

6.1.1 Setting up the Differential Equation

- How a population evolves exactly, depends on numerous factors such as temperature, food supply, the presence of predators, etc. We assume in the following that the temperature is kept constant, there is no shortage of food, and predators are absent. Thus, external parameters shall **not** affect the population.
- Under these conditions, the assumption is reasonable that the growth of the population depends solely on the number of individuals which already exist and the (constant) reproduction rate. We can write the increment ΔN within a time interval Δt as

$$\Delta N \propto N(t) \cdot \Delta t \tag{6.1}$$

growth constant k

- To replace the proportionality with an equation, we introduce a proportionality constant, the **growth constant** k, and we change to infinitesimal quantities (differentials) dN and dt,

$$dN = k \cdot N(t) \cdot dt \tag{6.2}$$

Formal division by dt yields

$$\boxed{\frac{dN}{dt} = k\, N(t)} \tag{6.3}$$

- Equation (6.3) relates a—still unknown—function $N(t)$ with its derivative. Such an equation is dubbed **differential equation**.
- Here we have the most elementary form, a **homogeneous, ordinary, linear differential equation of 1. order**. These terms have the following meaning. Homogeneous, all terms of the equation contain $N(t)$ or its derivative; ordinary, the unknown function depends only on one variable; linear, $N(t)$ and its derivative enter in the equation only linearly; 1. order, the highest derivative present is the first.
- Our task is to find the **solution $N(t)$** to the differential equation, i. e., a function which, when inserted, fulfills the equation.

6.1.2 Solution of the Differential Equation

initial conditions

- For solving a differential equation it is important to know the **initial or boundary conditions**. In our case, it is the number of individuals at time zero. We dub it N_0,

$$N(0) = N_0 \tag{6.4}$$

separation of the variables

- The solution of an ordinary differential equation of 1. order is easily obtained with the method of **separation of the variables**. To this end we move all factors of the dependent variable N to one side of the equation and the independent variable t to the other,

$$\frac{dN}{N} = k\,dt \tag{6.5}$$

- Now we can integrate both sides, from time zero to a later time t. We have to make sure that the values of N and t are correctly assigned to each other: At time zero, the number of individuals is N_0 according to the initial condition, at the later time t it is $N(t)$,

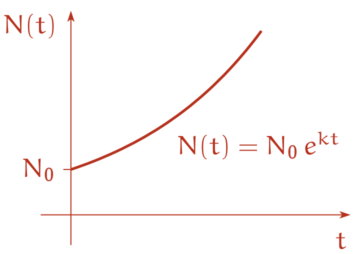
$$\int_{N_0}^{N(t)} \frac{dN^*}{N^*} = k \int_0^t dt^* \tag{6.6}$$

The integration variables have been renamed N^* and t^*, respectively, to use different symbols for variables and integration bounds.

- Performing the integrals yields

$$\ln N(t) - \ln N_0 = k\,t \tag{6.7}$$

$$\ln \frac{N(t)}{N_0} = k\,t \tag{6.8}$$



- Forming the antilogarithm of both sides yields the final result

$$\boxed{N(t) = N_0\, e^{kt}} \tag{6.9}$$

- The population grows without limit according to an exponential function. In the real world, the increase will eventually level off at some time, e. g., when the population runs out of food or when the living conditions become unfavorable for some other reason. In this case, additional terms must be added to the differential equation [Eq. (6.3)], so its solution changes.

6.2 Radioactive Decay

- The number of atomic nuclei of a radioactive substance decreases with time due to the disintegrations. Let us assume that N_0 nuclei are present at time $t = 0$. We wish to calculate their number $N(t)$ at some later time t.

- Also in this case it is reasonable to assume that the number of disintegrations within a given time interval is the higher, the more nuclei are still present,

$$\Delta N \propto -N(t) \cdot \Delta t \tag{6.10}$$

or, using differentials and a proportionality constant,

$$dN = -\lambda \cdot N(t) \cdot dt \tag{6.11}$$

decay constant λ

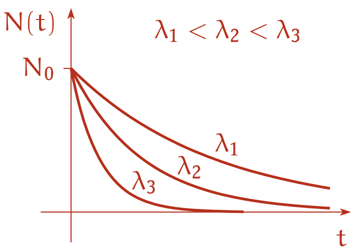
- The minus sign corresponds to the fact that ΔN or dN are **negative** due to the decay. The proportionality constant λ is then positive. It is dubbed the **decay constant** of the substance.
- The solution $N(t)$ can again be obtained via separation of the variables. It reads

$$\boxed{N(t) = N_0\, e^{-\lambda t}} \tag{6.12}$$

- The calculation and the result are very similar to the case of the population growth discussed above. The only difference is the negative sign in the exponent.

half-life $T_{1/2}$

- The decay constant is related to the **half-life** $T_{1/2}$ of the material. After one half-life, the number of nuclei has decayed to half its initial value, i. e.,



$$N\left(T_{1/2}\right) = \frac{1}{2} N_0 \tag{6.13}$$

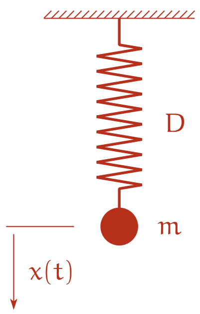
The relation reads

$$T_{1/2} = \frac{\ln 2}{\lambda} \tag{6.14}$$

The bigger λ or the shorter the half-life is, the faster the substance decays.

- These examples demonstrate the extraordinary importance of the **exponential function with base** e in nature: It describes **all phenomena in which the derivative of a function is proportional to the function itself**.

6.3 Harmonic Oscillation of a Spring Pendulum



- A spring pendulum is a mass of appropriate size (e. g., a little steel ball) suspended from a helical spring. Let the mass be m, the spring constant D.
- If we pull the mass down from its rest position and release it, it begins to oscillate about this position. Our goal is to derive the mathematical form of the oscillation and its period. For the sake of simplicity we neglect all friction effects.
- The **equation of motion** follows from the **balance of forces**: The restoring force of the elongated or compressed spring is equal to the force which accelerates the mass, so

$$m\, a(t) = -D\, x(t) \tag{6.15}$$

Here $x(t)$ is the displacement of the mass from its equilibrium position at time t and $a(t) = \ddot{x}(t) = d^2x/dt^2$ is its acceleration. We can rewrite the equation

$$\boxed{\frac{d^2x}{dt^2} = -\frac{D}{m}\,x(t)} \tag{6.16}$$

- This is a **differential equation of 2. order**. The method of separation of the variables fails to solve it. Instead we use an **ansatz**, i. e., we write down the presumed mathematical form of the solution but leave its parameters open for later determination.
- Oscillations are described by periodic functions. Hence, we choose the ansatz

ansatz for the solution

$$x(t) = x_0\, e^{i\omega t} \tag{6.17}$$

Its physically meaningful part is the **real part**. The derivatives read

$$\dot{x}(t) = i\omega\, x_0\, e^{i\omega t} \tag{6.18}$$

$$\ddot{x}(t) = -\omega^2\, x_0\, e^{i\omega t} \tag{6.19}$$

- Inserting x and $\ddot{x}$ in the differential equation [Eq. (6.16)] yields

$$-\omega^2\, x_0\, e^{i\omega t} = -\frac{D}{m}\, x_0\, e^{i\omega t} \tag{6.20}$$

and, thus,

$$\boxed{\omega = \sqrt{\frac{D}{m}} \qquad \text{or} \qquad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{D}}} \tag{6.21}$$

- The ansatz [Eq. (6.17)] indeed solves the differential equation. By inserting it and its second time derivative we obtain **the angular frequency (angular velocity) ω, the regular frequency $\nu = \omega/2\pi$, and the period $T = 1/\nu = 2\pi/\omega$** of the vibration. The amplitude (the maximum elongation of the spring) x_0 is arbitrary. It depends on the initial condition, i. e., on the elongation of the spring before releasing it.
- For the ansatz, we could have used the functions

$$x(t) = x_0\, e^{-i\omega t} \tag{6.22}$$

$$x(t) = x_0\, \sin(\omega t) \tag{6.23}$$

$$x(t) = x_0\, \cos(\omega t) \tag{6.24}$$

as well. They all yield the same vibration frequency. Complex exponentials are often more convenient for the calculation than trigonometric functions, in particular when a damping term describing friction effects is included in the differential equation. This point, however, is beyond the scope of the present tutorial. In any case, the physically meaningful part of a complex result is always the real part.